Complex Analysis Prelim

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January 2018

Instructions: Do as many of the following problems as you can. Four completely correct solutions will guarantee a PhD pass. A few completely correct solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill in the gap. You may use any standard theorem from the complex analysis course, identifying it either by name or stating it in full.

Notation: $\mathbb{C}$ is the complex plane and $\mathbb{D}$ is the open unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

1. Let $v : \mathbb{C} \setminus [0, \infty) \to (0, 2\pi)$ be the function $v(x + iy) = \arg(x + iy)$, $x, y \in \mathbb{R}$.
   ($\mathbb{C} \setminus [0, \infty)$ is the complement in $\mathbb{C}$ of $\{x + 0i : x \geq 0\}$)

   (a) Compute $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$.

   (b) Determine whether v is harmonic.

2. Suppose that f an entire function satisfying $\lim_{|z| \to \infty} |f(z)| = \infty$. Prove that f is a polynomial.

3. (a) Prove that $(z^2 - 1)^{-1}$ has an analytic square root on the domain $\mathbb{C} \setminus [-1, 1]$.
   ($\mathbb{C} \setminus [-1, 1]$ is the complement in $\mathbb{C}$ of the line segment from -1 to 1)

   (b) Find the Laurent expansion of an analytic square root from part (a) on the domain $\{z : |z| > 1\}$, centered about $z = 0$.

4. Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function. Assume that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$, and that f has a zero of order $m \geq 1$ at the origin. Show that $|f(z)| \leq |z|^m$.

5. Find a conformal mapping from $\mathbb{D} \cap \{z : \operatorname{Re} z > 0\}$ onto the wedge

$$\{z \in \mathbb{C} : 0 < \operatorname{Im} z < \operatorname{Re} z\}.$$

6. Suppose that f is analytic and one-to-one on a domain U. Prove that f' does not has no zeros on U.

7. Let $F(z) = \frac{\pi}{z^4} \frac{\cos(\pi z)}{\sin(\pi z)}$.

   (a) Find all poles of F and compute the residue of F at each pole.

   (b) For a positive integer N, let C_N be the rectangle with vertices $\pm(N + 1/2) \pm iN$, traversed counterclockwise once. Show that $\lim_{N \to \infty} \int_{C_N} F(z) dz = 0$.

   (c) Conclude from parts (a) and (b) that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.