MATH 5510 January 2018 PRELIMINARY EXAMINATION

INSTRUCTIONS: Solve three out of five questions. You do not have to prove results which you rely upon, just state them clearly.

Good luck!

Q1) Answer 3 out of 4 questions (a), (b), (c), (d).

(a) Let we are given $(n+1)$ points $\{x_k, y_k\}$, $k = 0, 1, \ldots, n$. Give a proof that the interpolation problem of finding a polynomial

$$P_{01\ldots n}(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n$$

whose degree does not exceed n and such that

$$P_{01\cdots n}(x_k) = y_k \qquad (k = 0, 1, \ldots, n),$$

is equivalent to solving a linear system

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & & \vdots & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}}_{V_{n+1}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

with a Vandermone coefficient matrix V_{n+1}.

(b) Derive the factorization

$$V_{n+1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & & \vdots \\ 1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & x_2 - x_1 & \ddots & & \vdots \\ \vdots & \ddots & x_3 - x_1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & x_n - x_1 \end{bmatrix} \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & V_n \end{array}\right] \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 0 & 1 & x_1 & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & x_1^2 \\ \vdots & & \ddots & \ddots & x_1 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

(c) Use the factorization of (b) to derive a recursive formula for the determinant of the Vandermonde matrix V_{n+1}. Use the latter to prove that the interpolation problem of (a) is always solvable and that the solution is unique.

(d) Show that if the function f has an $(n+1)$st derivative, then for every argument y there exists a number s in the smallest interval $I[x_0, ..., x_n, y]$ which contains y and support abscissas x_i, satisfying

$$f(y) - P_{01\ldots n}(y) = \frac{w(y) f^{(n+1)}(s)}{(n+1)!}$$

where $P_{01\ldots n}(y)$ is the interpolating polynomial

$$P_{01\cdots n}(x_j) = f(x_j) \qquad (j = 0, 1, \ldots, n),$$

and

$$w(x) = (x - x_0)(x - x_1) \cdots (x - x_n).$$

Q2) Answer 3 out of 3 questions (a), (b), (c).

(a) Prove that the Householder reflection matrix $P = I - 2ww^*$ (with $w^*w = 1$) is unitary and that $P^2 = I$.

(b) For a given vector x explain how to find w such that

$$Px = ke_1$$

with some k. Derive explicit formulas for w and k.

(c) Describe how, for a real matrix A, a sequence of Housholder reflections can be used to compute the QR factorization $A = QR$ with orthogonal Q and upper triangular R.

Q3) Answer 3 out of 4 questions (a), (b), (c), (d).

(a) Let $\|x\|$ denotes the usual Euclidean norm $\sqrt{x^T x}$. Prove that the linear least squares problem

$$\min_{x \in \mathbb{R}^n} \|y - Ax\|$$

with a $m \times n$ matrix A has at least one minimal point x_0.

(b) Prove that if x_1 is another minimum point, then $Ax_0 = Ax_1$. The residual $r := y - Ax$ is uniquely determined and satisfies the equation $A^T r = 0$.

(c) Prove that Every minimum point x_0 is also a solution of normal equations

$$A^T A x = A^T y$$

and conversely.

(d) Explain how the orthogonalization technique of Q2 (that is, computing for the $m \times n$ matrix A the factorization $A = QR$ with $m \times m$ orthogonal matrix Q and $m \times n$ upper triangular matrix R) yields an efficient algorithm for solving the above least squares problem.

Q3) Answer 3 out of 4 questions (a), (b), (c), (d).

(a) Let T be an $n \times n$ positive definite matrix. Relate the factorization

$$T\widetilde{U} = \widetilde{L} \tag{1}$$

to the standard LDL^* factorization of T to prove that (1) always exists and it is unique. Here $\widetilde{U}$ is a unit (i.e., with 1's on the main diagonal) upper triangular matrix, and $\widetilde{L}$ is a lower triangular matrix.

(b) Let $\langle \cdot, \cdot \rangle$ be an arbitrary inner product in the vector space Π_n (of all polynomials whose degree does not exceed n). Let T be a positive definite moment matrix, i.e., $T = [\langle x^i, x^j \rangle]_{i,j=0}^n$. Let

$$u_k(x) = u_{0,k} + u_{1,k}x + u_{2,k}x^2 + \ldots + u_{k-1,k}x^{k-1} + x^k. \tag{2}$$

be the k-th orthogonal polynomial with respect to $\langle \cdot, \cdot \rangle$. Prove that the k-th column of the matrix $\widetilde{U}$ of (a) contains the coefficients of $u_k(x)$ as in

$$\widetilde{U} = \begin{bmatrix} 1 & u_{0,1} & u_{0,2} & u_{0,3} & \cdots & \cdots & u_{0,n} \\ 0 & 1 & u_{1,2} & u_{1,3} & \cdots & \cdots & u_{1,n} \\ 0 & 0 & 1 & u_{2,3} & \cdots & \cdots & u_{2,n} \\ \vdots & & 0 & 1 & \cdots & \cdots & u_{3,n} \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & 1 & u_{n-1,n} \\ 0 & & & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

(c) Assuming now that the moment matrix T has Toeplitz structure derive the so-called Levinson algorithm, that is, an algorithm to compute the columns of $\widetilde{U}$ based on the formula (deduce it) that relates the k-th column u_k of U to its "predecessor" u_{k-1} ($k = 2, 3, \ldots, n$).

Hint: Use the fact (no need to prove it) that Toeplitz moment matrices T have the following property: if

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}$$

then

$$T \begin{bmatrix} x_n^* \\ x_{n-1}^* \\ x_{n-2}^* \\ \vdots \\ x_3^* \\ x_2^* \\ x_1^* \end{bmatrix} = \begin{bmatrix} y_n^* \\ y_{n-1}^* \\ y_{n-2}^* \\ \vdots \\ y_3^* \\ y_2^* \\ y_1^* \end{bmatrix}$$

(d) Prove that the algorithm of (c) uses $O(n^2)$ arithmetic operations.

Q5) Answer 4 out of 5 questions (a), (b), (c), (d), (e).

(a) Derive the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for the Chebyshev polynomials:

$$T_n(x) = \cos(n \cos^{-1} x), \quad n = 0, 1, \ldots.$$

and prove that $\hat{T}_n(x) = (1/2^{n-1})T_n(x)$ is a monic polynomial (that is, the leading coefficient is 1).

(b) Derive the formula for all the zeros of $T_n(x)$.

(c) Derive the formula for all the extrema of $T_n(x)$ in the closed interval $[-1, 1]$.

(d) Prove that $\hat{T}_n(x)$ has minimal infinity norm among all monic polynomials of degree n on the interval $[-1, 1]$. Specifically, show that $\|\hat{T}_n(x)\|_\infty = 1/2^{n-1}$, where $\|\cdot\|_\infty$ denotes the maximum norm of a function on the interval $[-1, 1]$.

(e) Prove that Chebyshev polynomials are orthogonal with respect to the inner product in Π_n defined by

$$< a(x), b(x) >= \int_{-1}^{1} \frac{a(x)b(x)}{\sqrt{1-x^2}} dx.$$