Abstract Algebra Prelim — Jan. 2019

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. (**10 pts**)
   (a) (**7 pts**) For $n \geq 3$, determine with proof the conjugacy classes of the dihedral group of order $2n$. (Hint: Separately consider even n and odd n.)
   (b) (**3 pts**) Let C_n be the number of conjugacy classes in the dihedral group of order $2n$. Compute $\lim_{n\to\infty} \frac{C_n}{n}$.

2. (**10 pts**) Let p the *smallest* prime dividing the order of a finite group G. Prove that if H is a subgroup of G with index p then H is a normal subgroup. (Hint: Look at the left multiplication action of G on the left cosets of H.)

3. (**10 pts**) View $\mathbf{Q}$ and $\mathbf{Z}$ as additive groups. For $a \in \mathbf{Z}$, set $\varphi_a \colon \mathbf{Q} \to \mathbf{Q}$ by $\varphi_a(t) = 2^a t$.
   (a) (**4 pts**) Show that φ_a is an automorphism of (the additive group) $\mathbf{Q}$ for each $a \in \mathbf{Z}$ and show $\varphi\colon \mathbf{Z} \to \mathrm{Aut}(\mathbf{Q})$ given by $a \mapsto \varphi_a$ is a homomorphism of groups.
   (b) (**4 pts**) Set $G = \mathbf{Q} \rtimes_\varphi \mathbf{Z}$, a semi-direct product. In G let $H = \{(m, 0) : m \in \mathbf{Z}\}$ and $x = (0, 1)$. Prove that $xHx^{-1} \subset H$.
   (c) (**2 pts**) Show that $x = (0, 1)$ is *not* an element of the normalizer $\mathrm{N}_G(H)$ of H in G.

4. (**10 pts**)
   (a) (**4 pts**) Define a Euclidean domain and prove all ideals in a Euclidean domain are principal.
   (b) (**4 pts**) Prove $F[X]$ is a Euclidean domain when F is a field.
   (c) (**2 pts**) Prove $\mathbf{Z}[X]$ is not a Euclidean domain.

5. (**10 pts**)
   (a) (**2 pts**) For a commutative ring R and R-module M, define what it means to say M is a cyclic R-module.
   (b) For any matrix $A \in \mathrm{M}_n(\mathbf{R})$, we can make $\mathbf{R}^n$ into an $\mathbf{R}[t]$-module by declaring that for any polynomial $f(t) = c_0 + c_1 t + \cdots + c_d t^d$ in $\mathbf{R}[t]$ and vector v in $\mathbf{R}^n$, $f(t)v = f(A)v = (c_0 I + c_1 A + \cdots + c_d A^d)v$.
   Determine, with explanation, whether $\mathbf{R}^n$ is a cyclic $\mathbf{R}[t]$-module for each of the following choices of A. If it is a cyclic $\mathbf{R}[t]$-module, then find an $\mathbf{R}[t]$-generator:
      i. (**4 pts**) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $\mathbf{R}^2$,
      ii. (**4 pts**) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ on $\mathbf{R}^3$.

6. (**10 pts**) Give examples as requested, with justification.
   (a) (**2.5 pts**) A group isomorphism from $(\mathbf{Z}/7\mathbf{Z})^\times$ to $(\mathbf{Z}/9\mathbf{Z})^\times$.
   (b) (**2.5 pts**) A cyclic group with 20 generators.
   (c) (**2.5 pts**) A unit in $\mathbf{Z}[\sqrt{11}]$ other than ± 1.
   (d) (**2.5 pts**) A prime element of $\mathbf{Z}[i]$.