Complex Functions Prelim, August 2019.

- $\mathbb{D}$ denotes the unit disk: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.
- When using a known result in your working **you must clearly state the result and make it clear that you have verified all required hypotheses.**

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1. (a) Find all functions f which are holomorphic on $\mathbb{C} \setminus \{0\}$ and have the property that $z^2 f(z)$ is bounded on $\mathbb{C} \setminus \{0\}$.

   (b) Find all functions f which are holomorphic on $\mathbb{C} \setminus \{0\}$ and have the property that $z \sin(z) f(z)$ is bounded on $\mathbb{C} \setminus \{0\}$.

2. Let $\gamma \subset \mathbb{C}$ be a positively-oriented simple closed curve not intersecting the set $\{-1, 1\}$. Compute all possible values of the integral

$$\int_\gamma \frac{2dz}{z^2 - 1},$$

and give examples of curves γ which realize each value.

3. State and prove the Schwarz Lemma, including what occurs in the case of equality.

4. (a) Show that a continuous function $f : \mathbb{D} \to \mathbb{C}$ that is holomorphic on the slit disc $\mathbb{D} \setminus [0, 1)$ is holomorphic on $\mathbb{D}$.

   (b) Give an example of a holomorphic function $f : \mathbb{D} \setminus [0, 1) \to \mathbb{C}$ that has no holomorphic extension to $\mathbb{D}$.

5. In this problem, $p_a(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$ is a cubic polynomial with coefficient vector $a = (a_0, a_1, a_2, a_3)$.

   (a) State Rouché's theorem.

   (b) The polynomial $p_{(1,1,1,1)}(z) = 1+z+z^2+z^3$ has a simple root at $z = -1$. Without using an explict solution of the cubic, show that there is a neighborhood $U \subset \mathbb{C}^4$ of $(1,1,1,1)$ such that if $a \in U$ then $p_a(z)$ has a unique root $r(a)$ close to -1.

   (c) Show that if the neighborhood U is sufficiently small then $a \mapsto r(a)$ is a continuous function on U.

6. Let $u : \overline{\mathbb{D}} \to \mathbb{R}$ be a positive and continuous function on the closed unit disc which is harmonic on $\mathbb{D}$. If $K \subset \mathbb{D}$ is compact, show that there is C (that may depend on K) such that

$$\max_K u(x, y) \leq C \min_K u(x, y).$$

7. Suppose f_n is a sequence of holomorphic functions on $\mathbb{D}$ such that $\text{Re}(f_n(z)) > 0$ for all $z \in \mathbb{D}$ and all n.

   (a) If $f_n(0) = 1$ for all n, show that f_n has a subsequence that converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic f for which $\text{Re}(f(z)) > 0$ on $\mathbb{D}$.

   (b) Is this true without the assumption that $f_n(0) = 1$ for all n?