Real Analysis Preliminary Exam, August 2019

Instructions and notation:

(i) Complete all problems. Give full justifications for all answers in the exam booklet.

(ii) Lebesgue measure on $\mathbb{R}^n$ is denoted by m or dx. The σ-algebra of Borel sets in $\mathbb{R}^n$ is denoted by $\mathcal{B}(\mathbb{R}^n)$. The characteristic function of a set A is denoted by χ_A. If $S \subset \mathbb{R}^n$ is Lebesgue measurable then $L^p(S) := L^p(m|_S)$ where $m|_S$ is the restriction of the Lebesgue measure m on S.

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1. (10 points) Let $0 < a < b < \infty$, and consider the function

$$f(x) = \frac{1}{x^a + x^b}, \quad x > 0.$$

For what values of p does $f \in L^p((0, \infty))$?

2. (10 points) Let $g : [0, 1] \to \mathbb{R}$ be a nonnegative Lebesgue measurable function.
   (a) Prove that, as $n \to \infty$, the numbers $I_n = \int_{[0,1]} g^n \, dm$ converge to a non-negative limit that may be infinite.
   (b) If $I_n = C < \infty$ for all $n \in \mathbb{N}$, show that there exists some Lebesgue measurable set $A \subset [0, 1]$ such that $g = \chi_A$, m-a.e.

3. (10 points) Let μ, ν be two measures on a measurable space $(X, \mathcal{A})$. Prove that if for every $\epsilon > 0$ there exists a measurable set A_ϵ such that $\mu(A_\epsilon) < \epsilon$ and $\nu(A_\epsilon^c) < \epsilon$ then $\mu \perp \nu$.

4. (15 points) Prove or disprove three of the following statements.
   (a) If $(f_n)_{n \in \mathbb{N}}$, $f_n : \mathbb{R} \to \mathbb{R}$, is a sequence of measurable functions such that $f_n \to 0$ in $L^3(\mathbb{R})$ and in $L^5(\mathbb{R})$ then $f_n \to 0$ in $L^4(\mathbb{R})$.
   (b) There exists a function $f : [0, 1] \to \mathbb{R}$ such that $f(0) = 0$, $f(1) = 1$ and

$$\sup_{x \neq y, |x-y|<1} \frac{|f(x) - f(y)|}{|x - y|} < 1.$$

   (c) There exists a measurable set $A \subset [0, 1]$ with $m(A) = 0.9$ such that $m(A \cap I) > 0.1\, m(I)$ for every open interval $I \subset [0, 1]$.
   (d) There exists a probability measure μ on $\mathcal{B}(\mathbb{R}^n)$ such that $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}^n$ and $\mu(B) \in \{0, 1\}$ for $B \in \mathcal{B}(\mathbb{R}^n)$.

5. (10 points) Let $A \subset \mathbb{R}^n$ be a Lebesgue measurable set such that $m(A) < \infty$ and let $t \in (0, m(A)/2)$. Prove that there exist disjoint Lebesgue measurable sets $B, C \subset A$ such that $m(B) = m(C) = t$.

6. (10 points) Prove that $L^\infty([0, 1])$ is a set of first category in the space $(L^1([0, 1]), \| \cdot \|_1)$.

   Hint: Consider the sets $E_n = \{f \in L^\infty([0, 1]) : \|f\|_\infty \leq n\}$.

   Recall that a subset of a topological space X is of *first category* if it can be expressed as the union of countably many nowhere dense subsets of X. A set $A \subset X$ is *nowhere dense* if for any open set V there exists an open set $U \subset V$ such that $U \cap A = \emptyset$.