Abstract Algebra Prelim Jan. 2020

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. (**10 pts**) Let p be a prime number.
   (a) (**5 pts**) Show every group of order p^n where $n \geq 1$ has a nontrivial center.
   (b) (**5 pts**) Use part (a) to show every group whose order is p^2 is abelian.

2. (**10 pts**) For $a \in \mathbf{Z}$ and $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{R}^3$, define
$$a * \mathbf{u} = (u_1, au_1 + u_2, a^2u_1 + 2au_2 + u_3).$$
   (a) (**7 pts**) Prove the above formula defines an action of the *additive* group $(\mathbf{Z}, +)$ on $\mathbf{R}^3$.
   (b) (**3 pts**) Show a vector $\mathbf{u} = (u_1, u_2, u_3)$ in $\mathbf{R}^3$ has a finite $\mathbf{Z}$-orbit for this action if and only if $u_1 = u_2 = 0$.

3. (**10 pts**) The goal of this problem is to classify all groups of order 35 up to isomorphism.
   (a) (**4 pts**) Determine all *abelian* groups of order 35 up to isomorphism.
   (b) (**6 pts**) Show that every group of order 35 is abelian.

4. (**10 pts**) Let I be the ideal $(7, 1 + \sqrt{-13})$ in $\mathbf{Z}[\sqrt{-13}]$.
   (a) (**5 pts**) Show the ring homomorphism $\mathbf{Z}/7\mathbf{Z} \to \mathbf{Z}[\sqrt{-13}]/I$ given by $a \bmod 7\mathbf{Z} \mapsto a \bmod I$ is an isomorphism.
   (b) (**5 pts**) Show I is *not* principal.

5. (**10 pts**) Let p be a prime number.
   (a) (**3 pts**) Prove $\mathbf{Z}[x]/p\mathbf{Z}[x] \cong (\mathbf{Z}/p\mathbf{Z})[x]$ as rings.
   (b) (**7 pts**) Prove that a maximal ideal in $\mathbf{Z}[x]$ that contains p must have the form $(p, f(x))$ where $f(x)$ is monic in $\mathbf{Z}[x]$ and $f(x) \bmod p$ is irreducible in $(\mathbf{Z}/p\mathbf{Z})[x]$.

6. (**10 pts**) Give examples as requested, with justification.
   (a) (**2.5 pts**) A nonabelian group of order 21.
   (b) (**2.5 pts**) An expression of (12345) as a product of transpositions.
   (c) (**2.5 pts**) Gaussian integers γ and ρ such that $7+2i = (2+3i)\gamma + \rho$ and $\mathrm{N}(\rho) < \mathrm{N}(2+3i)$.
   (d) (**2.5 pts**) A homomorphism of commutative rings $f\colon R \to S$ and an ideal I in R such that $f(I)$ is not an ideal in S.