Geometry and Topology Exam

January 2020

Problem 1. Let X be a topological space. Prove or disprove the following assertions.

1. Let $A_1, \ldots, A_k$ be subsets of X. Then $\overline{\bigcup_{i=1}^k A_i} = \bigcup_{i=1}^k \overline{A_i}$.
2. Let $\{B_i\}_{i=1}^\infty$ be subsets of X. Then $\overline{\bigcup_{i=1}^\infty B_i} = \bigcup_{i=1}^\infty \overline{B_i}$.

Problem 2. Let (X, d) be a complete metric space and $\{E_i\}_{i=1}^\infty$ be a sequence of nonempty closed subsets so that $E_{i+1} \subseteq E_i$ for all i. Suppose the diameter $\operatorname{diam}(E_i) \to 0$ as $i \to \infty$. Show that $\bigcap_{i=1}^\infty E_i$ is nonempty and consists of precisely one point. (Recall that the diameter of a metric space E is defined by $\operatorname{diam}(E) = \sup\{d(x, y) : x, y \in E\}$.)

Problem 3. Let $\mathcal{Z}$ be the topology on $\mathbb{R}^2$ such that every nonempty open set of $\mathcal{Z}$ is of the form $\mathbb{R}^2 \setminus \{\text{at most finitely many points}\}$. Show that any continuous function $f : (\mathbb{R}^2, \mathcal{Z}) \to \mathbb{R}$ is constant, where $\mathbb{R}$ is endowed with the standard topology.

Problem 4. Let E, X be topological spaces and $q : E \to X$ be a covering map. Show that the following statements are equivalent.

1. E is compact.
2. X is compact and the fiber $q^{-1}(x)$ is finite for all $x \in X$.

Problem 5. Let D^2 be a closed disk in $\mathbb{R}^2$ and S^1 be the boundary unit circle. Prove or disprove the following statements.

1. Let $f : S^1 \to D^2$ be a continuous map. Then f extends to a continuous map $F : D^2 \to D^2$ with $F|_{S^1} = f$.
2. There is a map $g : D^2 \to S^1$ such that $g|_{S^1}$ is the identity map on S^1.

Problem 6. Let X, Y, Z be convex open subsets in $\mathbb{R}^n$, $n \geq 1$. Suppose $X \cap Y \cap Z \neq \emptyset$. Show that their union $X \cup Y \cup Z$ is simply-connected.