Risk Theory Prelims for Actuarial Students
Monday, 20 January 2020

Instructions:

1. There are five (5) questions here and you are to answer all five. Each question is worth 20 points.
2. Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
3. Please write legibly. Points will be deducted for incoherent, incorrect, and/or irrelevant statements.

Question No. 1:

Consider an insurer whose aggregate risk is given by the following collective risk model:

$$S = \sum_{i=1}^{N} X_i,$$

where N follows a Poisson distribution with mean 100, and X_i's are i.i.d. random variables that follow an exponential distribution with mean 100. Assume N and X_i's are independent.

The insurer charges premium based on the expected value principle with loading α, i.e. $P = (1+\alpha)\,\mathrm{E}[S]$. To manage risk, the insurer uses proportional reinsurance contracts with retention proportion β, i.e. the insurer pays $S_I = \sum_{i=1}^{N} Y_i$, where $Y_i = \beta X_i$. The cost of proportional reinsurance is also calculated using the expected value principle but with loading $\alpha_R = 30\%$.

Denote by P_R the total premium paid by the insurer to the reinsurer. The insurer's Profit & Loss (P&L) is then obtained by

$$W = P - P_R - S_I.$$

The insurer would like to achieve 95% profit probability ($\Pr(W > 0) = 95\%$).

(a) If the insurer sets $\beta = 1$ (no reinsurance), what is α?

(b) If the insurer sets $\alpha = 25\%$, what is β?

Note. Please apply **normal approximations** when computing profit probability.

Question No. 2:

In a Bayesian framework, an unknown parameter θ is estimated by first specifying a prior distribution in terms of say the density or mass function $f(\theta)$ and then using the posterior distribution about θ once we have an observed sample $\mathbf{x}$.

Any reasonable estimate of θ is a function of this sample so that we may write $\widehat{\theta} = g(\mathbf{x})$. The loss or penalty to pay as a result of this estimation is usually expressed as a loss function given by

$$\ell(\theta, \widehat{\theta}) = \ell(\theta, g(\mathbf{x})).$$

The decision problem is to find the Bayesian estimator of θ to be that value $\widehat{\theta} = g(\mathbf{x})$ which minimizes the expectation of the loss function with respect to the posterior about θ.

(a) Based on the squared error loss function

$$\ell(\theta, \widehat{\theta}) = (\theta - \widehat{\theta})^2,$$

prove that the Bayesian estimator is $\mathrm{E}[\Theta|\mathbf{x}]$.

(b) Based on the absolute value loss function

$$\ell(\theta, \widehat{\theta}) = |\theta - \widehat{\theta}|,$$

prove that the Bayesian estimator is the median of the distribution of $\Theta|\mathbf{x}$.

Question No. 3:

Suppose you are given data on annual aggregate claims (in 000's) for each of three employer liability risks over a period of five years in Table 1. Here, Y_{it} denotes claims in year t for risk (or employer) i.

Table 1: Annual aggregate claims for employer liability risks.

			Year				
Risk	1	2	3	4	5	$\bar{Y}_i$	$\sum_{t=1}^{5}(Y_{it} - \bar{Y}_i)^2$
1	3,894	5,188	3,582	4,680	5,182	4,505	2,180,690
2	3,940	2,994	3,582	4,068	4,434	3,804	1,190,476
3	4,382	5,028	4,434	4,844	5,642	4,866	1,049,784

(a) Using empirical Bayes method, calculate the credibility factor and credibility premium for the sixth year for each of the three risks.

(b) Now suppose you also have a risk volume, denoted by n_{it}, corresponding to each Y_{it} above. Some summary statistics for the risk volumes are as follows:

Table 2: Risk volume

Risk	$\sum_{t=1}^{5} n_{it}$
1	3,560
2	2,276
3	4,012

The credibility premium per unit of risk volume for the sixth year for risk 1 has been calculated to be 6.46. Calculate the credibility premium per unit of risk volume for the sixth year for risks 2 and 3.

Question No. 4:

Suppose the aggregate loss S for a pool of insurance contracts is given by

$$S = \sum_{i=1}^{N} X_i$$

where N is the claim frequency and X_i are the claim severities. Assume that:

- N has a Poisson distribution with mean $\lambda = 100$ and
- X_i has an exponential distribution with mean $\alpha = 10$ for $i = 1, 2, \ldots$.

Now all the policies in the pool are modified with a deductible of $d = 1$ and a maximum covered loss of $u = 100$. Denote the aggregate loss after modificiations by

$$\tilde{S} = \sum_{i=1}^{\tilde{N}} \tilde{X}_i$$

where $\tilde{N}$ is the claim frequency after modifications and $\tilde{X}_i$ are the claim severities after modifications.

(a) Show that $\tilde{N}$ has a Poisson distribution and calculate its mean $\tilde{\lambda} = \mathrm{E}[\tilde{N}]$.

(b) Calculate the mean and second moment of $\tilde{X}$: $\mathrm{E}[\tilde{X}]$ and $\mathrm{E}[\tilde{X}^2]$

(c) Calculate the mean and variance of $\tilde{S}$: $\mathrm{E}[\tilde{S}]$ and $\mathrm{Var}[\tilde{S}]$

Question No. 5:

Suppose the claim severity X has the following density function:

$$f_X(x) = 0.02\,x, \qquad 0 \le x \le c.$$

(a) Find the constant c.

(b) Calculate the expected value of X.

(c) Calculate the Value-of-Risk (VaR) of X at 81% level, $\mathrm{VaR}_{0.81}(X)$.

(d) Calculate the Tail-Value-of-Risk (TVaR) of X at 81% level, $\mathrm{TVaR}_{0.81}(X)$.

—— end of exam ——

APPENDIX

A random variable X is said to have a Gamma distribution with scale parameter $a > 0$ if its density has the form

$$f(x) = \frac{a^b x^{b-1} e^{-ax}}{\Gamma(b)}, \quad \text{for } x > 0.$$

A random variable X is said to be Pareto(α, θ) if its cumulative distribution function is expressed as

$$F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}.$$

A discrete random variable N is said to belong to the $(a, b, 0)$ class of distributions if it satisfies the relation

$$\Pr(N = k) = p_k = \left(a + \frac{b}{k}\right) \cdot p_{k-1}, \quad \text{for } k = 1, 2, \ldots,$$

for some constants a and b. The initial value p_0 is determined so that $\sum_{k=0}^{\infty} p_k = 1$.