Hilbert Spaces

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Inner Product

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Definition

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The **norm**, $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$, induced by an inner product defines a metric $d(x, y) = \|x - y\|$ on vector space $\mathscr{X}$.

Hilbert Space

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A **Hilbert space** is a vector space $\mathscr{H}$ over $\mathbb{F}$ with an inner product $\langle \cdot, \cdot \rangle$ such that relative to the metric induced by the norm, $\mathscr{H}$ is a complete metric space.

Hilbert Space

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Example

Let I be a set, and let $\ell^2(I)$ be the set of all functions $x : I \to \mathbb{F}$ such that $x(i) = 0$ for all but a countable number of i and

$$\sum_{i \in I} |x(i)|^2 < \infty.$$

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Then $\ell^2(I)$ is a Hilbert space.

Bounded Linear Functionals

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Proposition

A linear functional is bounded if and only if it is continuous.

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- Let $L : \mathscr{H} \to \mathbb{F}$ be a bounded linear functional. Let $v, h \in \mathscr{H}$ such that $h \neq 0$.

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- Then
$$|L(v+h) - L(v)| = |L(h)| \leq c|h|$$
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- Therefore L is Lipschitz continuous.
- Lipschitz continuity implies continuity.

$\square$

The Riesz Representation Theorem

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Theroem

If $L : \mathscr{H} \to \mathbb{F}$ *is a bounded linear functional, then there is a unique vector* h_0 *in* $\mathscr{H}$ *such that* $L(h) = \langle h, h_0 \rangle$ *for every* h *in* $\mathscr{H}$ *. Moreover,* $\|L\| = \|h_0\|$.

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$L^2[a,b] = \{f : [a,b] \to \mathbb{F} \mid \int_a^b |f(t)|^2 dt < \infty\}$ *is a Hilbert space with inner product given by*

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Corollary

If $F : L^2[a,b] \to \mathbb{F}$ is a bounded linear functional, then there is a unique h_0 in $Ł^2[a,b]$ such that

$$F(h) = \int h\overline{h_0}dt$$

for all h in $L^2[a,b]$.

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(a) if $e_1, e_2 \in \mathscr{E}$ and $e_1 \neq e_2$, then $e_1 \perp e_2$ for $e \in \mathscr{E}$, $\|e\| = 1$

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Definition

A **basis** of $\mathscr{H}$ is a maximal orthonormal set.

Bases

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Example

Let $\mathscr{H} = \ell^2(i)$ as before. For each $i \in I$, define e_i in $\mathscr{H}$ by $e_i(i) = 1$ and $e_j(j) = 0$ for $i \neq j$. Then $\{e_i \mid i \in I\}$ is a basis.

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Example

Let $\mathscr{H} = L^2_{\mathbb{C}}[0, 2\pi]$. For $n \in \mathbb{Z}$ define $e_n \in \mathscr{H}$ by $e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$. Then $\{e_n \mid n \in \mathbb{Z}\}$ is not only an orthonormal set, but also a basis for $\mathscr{H}$.

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- Consider the space $\ell^2(\mathbb{N})$, with basis $\{(1, 0, 0, \ldots), (0, 1, 0, 0, \ldots), (0, 0, 1, 0, \ldots), \ldots\}$.

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- Consider the space $\ell^2(\mathbb{N})$, with basis $\{(1,0,0,\dots),(0,1,0,0,\dots),(0,0,1,0,\dots),\dots\}$.
- We know
$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}<\infty.$$

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- So $(\frac{1}{1},\frac{1}{4},\frac{1}{9},\ldots)\in\ell^2(\mathbb{N})$.

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- We know
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty.$$
- So $(\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \ldots) \in \ell^2(\mathbb{N})$.
- But this element can't be written as a finite sum of basis elements.

Isomorphisms between Hilbert spaces

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If $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces, an **isomorphism** between $\mathscr{H}$ and $\mathscr{K}$ is a linear surjection $U : \mathscr{H} \to \mathscr{K}$ such that

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for all h, g in $\mathscr{H}$.

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If linear map U is an isomorphism, then U is an isometry.

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- Since $|\mathscr{E}| = |\mathscr{F}|$, $\ell^2(\mathscr{E})$ must be isomorphic to $\ell^2(\mathscr{F})$.

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- Do the same for $\mathscr{K}$ to $\ell^2(\mathscr{F})$.
- Since $|\mathscr{E}| = |\mathscr{F}|$, $\ell^2(\mathscr{E})$ must be isomorphic to $\ell^2(\mathscr{F})$.
- Conclude $\mathscr{H}$ is isomorphic to $\mathscr{K}$.

□

Further Topics Covered

- Linear Operators on Hibert spaces
- Fourier series
- Sturm-Liouville systems

Texts Used

- *A Course in Functional Analysis* by John B. Conway
- *An Introduction to Hilbert Spaces* based on the notes of Rodica D. Costin
- *Introduction to Partial Differential Equations and Hilbert Space Methods* by Karl E. Gustafson
- Ben Russo's brain