An introduction to ordinal numbers:

an excerpt from my DRP with Noah Hughes

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Dec. 7, 2016

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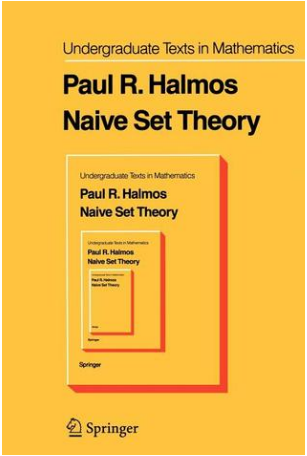
DRP Seminar

My book



A pack of wolves, a bunch of grapes or a flock of pigeons are all examples of sets of things.

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If an ordered pair (a, b) is an element of a relation R, we write aRb.
Relations are almost always denoted by symbols; one prominent example being the equivalence relation $=$.

Ordering

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Examples: $(\mathbb{N}, \leq)$, $(\mathbb{Q}, >)$, $(\wp(A), \subseteq)$.

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Definition: We call a set A with an order $\leq$ partially ordered if for every x, y, and z in A, we have

$$x \leq x;$$

$$x \leq y \text{ and } y \leq z \text{ implies } x \leq z;$$

$$x \leq y \text{ and } y \leq x \text{ implies } x = y;$$

and totally ordered if in addition for every $x, y \in A$ either

$$x \leq y \qquad \text{or} \qquad y \leq x.$$

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Why? Consider the set of negative integers.

We can, however, define a new order, $\leq_w$, that is a well ordering of the integers; we proceed to do so as an exercise in well ordering.

Example: Well Ordering of the Integers

Define $\leq_w$ such that, for all integers x, y, and z,

- if $|x| < |y|$, then $x \leq_w y$ (and vice-versa);
- if $|x| = |y|$, then

  if $x < y$, then $x <_w y$ (and vice-versa);

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Proof. Too long and technical for the scope of this presentation.
We will see its impact soon, but first we turn to constructing ordinals.

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We denote this set as ω, though you may know it as $\mathbb{N}$.

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Note: For finite ordinals, $x < y$ implies $x \in y$ and $x \leq y$ implies $x \subseteq y$.

Transfinite Ordinals

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ω is the first transfinite ordinal as well as the first limit ordinal.

Definition: A limit ordinal is any ordinal that has no immediate predecessor.

Transfinite Ordinals, continued

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Comparing to our well orderings of $\mathbb{Z}$

Recall:

$$(\mathbb{Z}, \leq_w) \cong 0, -1, 1, -2, 2, ...$$

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Closing Thoughts

- Ordinal arithmetic
- Cardinal numbers and arithmetic
- Continuum hypothesis

References

[1] Paul R. Halmos, *Naive set theory*, The University Series in Undergraduate Mathematics, D. Van Nostrand Co., Princeton, N.J.-Toronto-London-New York, 1960. MR0114756

Thank you!