GEOMETRY & TOPOLOGY PRELIM EXAM

AUGUST 2020

In the following problems, $\mathbb{R}^n$ and $\mathbb{C}$ are endowed with the standard topology unless otherwise indicated.

Problem 1. Prove or disprove the following statements.

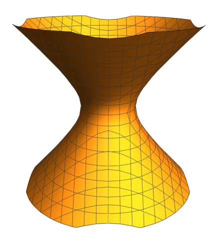
(1) Let A, B be two open subsets in a topological space X. Suppose that $A \bigcup B$ and $A \bigcap B$ are connected. Then A and B must be connected.

(2) Let $\{A_i\}$ be a countable collection of open subsets of a topological space X. Suppose that $\bigcup_i A_i$ and $\bigcap_i A_i$ are connected. Then A_i must be connected for each i.

Problem 2. Let $n \geq 2$. Define a topology $\mathcal{Z}$ on $\mathbb{R}^n$ such that every nonempty open set of $\mathcal{Z}$ is of the form $\mathbb{R}^n \setminus \{\text{at most finitely many points}\}$. Show that any continuous function $f : (\mathbb{R}^n, \mathcal{Z}) \to \mathbb{R}$ is constant.

Problem 3.
Let $f : X \to Y$ be a continuous and injective map between topological spaces X and Y. Prove that if X is compact and Y is Hausdorff, then f is an embedding.

Problem 4. Define $M = \{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2 - 1\}$ with the induced topology from $\mathbb{R}^3$.



(1) Find a universal covering of M.

(2) Let $X = M/\sim$ be the quotient space where $\sim$ is the equivalence relation generated by the relation $(x, y, z) \sim (x, y, -z)$. Is the quotient map $q : M \to X$ a covering map? Explain your answer.

Problem 5. Let S^1 denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, and let D denote the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$. Let X be a topological space. Prove that the following statements are equivalent:

(1) For every point $x \in X$, the fundamental group $\pi_1(X, x)$ is trivial.

(2) For every continuous function $f : S^1 \to X$, there exists a continuous map $F : D \to X$ which extends f.

Problem 6. Let X be the space obtaining from $\mathbb{R}^3$ by removing two circles $C_1 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + (y-1)^2 = 1, z = 0\}$ and $C_2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + (y+1)^2 = 1, z = 0\}$. Compute $\pi_1(X)$ and justify your answer.