Risk Theory Prelims for Actuarial Students
August 2020

Instructions:

1. There are five (5) questions here and you are to answer all five. Each question is worth 20 points.
2. Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
3. Please write legibly. Points will be deducted for incoherent, incorrect, and/or irrelevant statements.

Question No. 1:

Suppose aggregate loss S is given by a collective risk model, where the primary distribution is a Poisson $\mathcal{PN}(10)$ and the secondary distribution is given by

x	$\Pr(X = x)$
5	0.6
c	0.4

where $c > 5$ is a constant.

(a) Find the expected value and variance of S, $\mathrm{E}(S)$ and $\mathrm{Var}(S)$, respectively. You may use c to express the results.

(b) If $\mathrm{E}(S) = \mathrm{Var}(S)$, determine the value of c.

(c) Consider a stop-loss insurance written on S with deductible $d = 5$. If the expected loss after the deductible modification is 28, determine the value of c.

Question No. 2:

Individual loss amount X follows a two-parameter Pareto(α, θ) distribution with mean 25 and variance 3750. An insurance policy on X has a deductible amount of 5 and a policy limit of 100 per loss.
Assume loss amount increased due to inflation by 10% uniformly.

(a) Determine the parameters, α and θ, of this Pareto distribution.

(b) Calculate the expected value of claims per loss after the inflation.

(c) Calculate the variance of claims per loss after the inflation.

(d) Determine the effect of inflation for the coverage modification. The effect is defined to be

$$\text{Effect} = \frac{\text{EV(Claims) per Loss after Inflation}}{\text{EV(Claims) per Loss before Inflation}} - 1.$$

Question No. 3:

Suppose $\{X_i\,;\, i = 1, 2, \cdots, n\}$ is a sequence of i.i.d. random variables and each X_i follows an exponential distribution with density function given by

$$f(x) = \lambda\, e^{-\lambda x}, \quad \text{for } x > 0, \lambda > 0.$$

Define the maximal order statistics M_n by

$$M_n = \max(X_1, X_2, \cdots, X_n)\,.$$

(a) Find the cumulative distribution function (cdf) of M_n.

(b) Find the probability density function of M_n.

(c) Let $n = 3$, find the expectation of M_n in terms of λ.

(d) Define the sequence of constants $c_n := 1/\lambda$ and $d_n := \log(n)/\lambda$, where log is the natural logarithm. Determine the limiting distribution of

$$\frac{M_n - d_n}{c_n},$$

that is, by finding the corresponding cumulative distribution function, $H(\cdot)$, as solution to the following

$$H(x) = \lim_{n\to\infty} \Pr\left(\frac{M_n - d_n}{c_n} \le x\right).$$

Hint: $H(x)$ should be independent of λ, that is, free of the parameter λ.

Question No. 4:

Suppose X and Y are independent and identically distributed random variables, both following a Bernoulli distribution with parameter 0.02 (i.e., $\Pr(X = 1) = \Pr(Y = 1) = 0.02$).

(a) Compute $\mathrm{VaR}_{0.975}(X)$ and $\mathrm{VaR}_{0.975}(X + Y)$. Use the results to discuss which of the four axiom(s) in the definition of "coherent risk measure" the VaR does not satisfy.

(b) Compute $\mathrm{CVaR}_\delta(X)$ and $\mathrm{CVaR}_\delta(X + Y)$ for all $\delta \in (0, 1)$.

(c) Show that the axiom of "Subadditivity" holds true for the CVaR.

Remark. For a random variance X with cumulative distribution function (CDF) given by F_X, the Value-at-Risk (VaR) of X at probability level δ is defined by

$$\mathrm{VaR}_\delta(X) := \inf\{x \in \mathbb{R} : F_X(x) \ge \delta\}, \qquad \delta \in (0, 1),$$

and the conditional VaR (CVaR) of X at probability level δ is defined by

$$\mathrm{CVaR}_\delta(X) := \frac{1}{1-\delta}\int_\delta^1 \mathrm{VaR}_\xi\, d\xi, \qquad \delta \in (0, 1).$$

Question No. 5:

Ten observed values $x_1, x_2, \ldots, x_{10}$ are drawn from a mixture distribution with density function:

$$f(x) = \frac{2}{3} \cdot \frac{1}{\theta_1} e^{-x/\theta_1} + \frac{1}{3} \cdot \frac{1}{\theta_2} e^{-x/\theta_2}, \text{ for } x > 0$$

You are given the following additional information:

$$\sum_{i=1}^{10} x_i = 300 \quad \text{and} \quad \sum_{i=1}^{10} x_i^2 = 50,000$$

(a) Using the method of moments, estimate θ_1 and θ_2.

(b) Using the method of maximum likelihood, write the log of the likelihood given the observations $x_1, x_2, \ldots, x_{10}$.

(c) Using the method of maximum likelihood, find the two equations to solve for the parameter estimates for θ_1 and θ_2. DO NOT SOLVE.

—— end of exam ——

APPENDIX

A random variable X is said to have a Gamma distribution with scale parameter $a > 0$ if its density has the form

$$f(x) = \frac{a^b x^{b-1} e^{-ax}}{\Gamma(b)}, \quad \text{for } x > 0.$$

A random variable X is said to be a two-parameter Pareto(α, θ) if its cumulative distribution function has the form

$$F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}, \quad \text{for } x > 0.$$

It mean and variance are, respectively,

$$\mathrm{E}(x) = \frac{\theta}{\alpha - 1} \quad \text{and} \quad \mathrm{Var}(X) = \frac{\alpha\theta^2}{(\alpha-1)(\alpha-2)},$$

provided they exist.