Computability Theory

Jose Emilio Alcantara Regio & Waseet Kazmi

University of Connecticut

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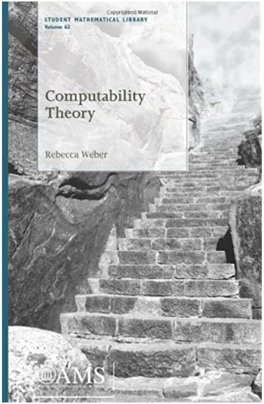
Materials used



Computability Theory, Rebecca Weber

Overview

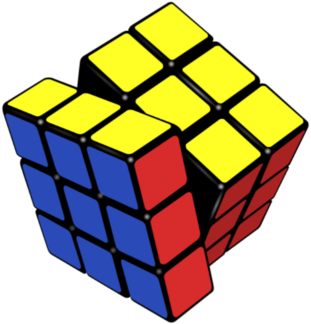
1. Introduction
2. Capturing Computability
3. Computable Functions
4. Computable and Computably Enumerable Sets
5. Turing Reductions
6. Turing Degrees

Introduction

Introduction

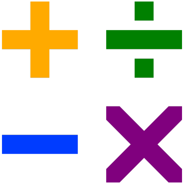
- What is Computability Theory?
- What does it mean to be *computable*?

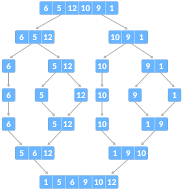
Examples of computability

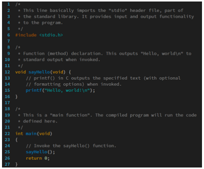


- There is an *algorithm*
- We can solve it in a finite amount of time using a finite number of *steps*

Examples of computability

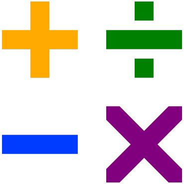


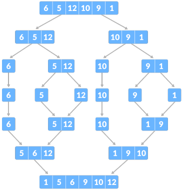


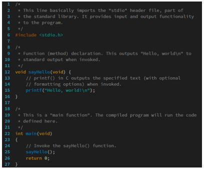


- Procedure
- Step-by-step

Examples of computability



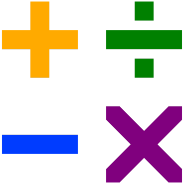


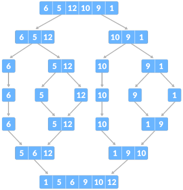


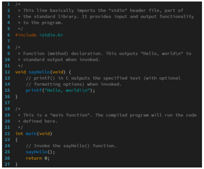
```c
/*
 * This line basically imports the "stdio" header file, part of
 * the standard library. It provides input and output functionality
 * to the program.
 */
#include <stdio.h>

/*
 * Function (method) declaration. This outputs "Hello, world\n" to
 * standard output when invoked.
 */
void sayHello(void) {
    // printf() in C outputs the specified text (with optional
    // formatting options) when invoked.
    printf("Hello, world!\n");
}

/*
 * This is a "main function". The compiled program will run the code
 * defined here.
 */
int main(void)
{
    // Invoke the sayHello() function.
    sayHello();
    return 0;
}
```

- “Procedure”?
- Step-by-step

Examples of computability







- “Procedure”?
- “Step-by-step”?

Defining Computability

- We need a **rigorous** definition for computability
- Must capture the intuitive understanding that we already have
- This was the goal of David Hilbert, Stephen Kleene, Alonzo Church, and Alan Turing
- Turing Machines were ultimately accepted as the satisfactory model for computation
- But why?

Capturing Computability

Preliminaries

Definition

A **partial function** is a function whose domain is a subset of $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Ex: $f(x) = \frac{1}{x}$, $f(x) = \log(x)$

Definition

A **total function** is a function whose domain is the entirety of $\mathbb{N}$.

- Why do we need partiality for functions?

$\Rightarrow$ The function might not be defined on some inputs

$\Rightarrow$ Or, the computation of the function on an input might never stop

Preliminaries

Definition

If x is in the domain of f, then we say that the computation of f on x **halts** or **converges**, denoted by $f(x)\downarrow$.

Definition

If x is not in the domain of f, then we say that the computation of f on x **diverges**, denoted by $f(x)\uparrow$.

Preliminaries

Definition

The **characteristic function** of a set A is a *total* function defined as follows:

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Some attempts at defining computability

- Partial recursive functions
  - Stephen Kleene
  - Purely mathematical intuition
- Lambda calculus
  - Alonzo Church
  - Substitution
  - Used today in functional programming languages such as Haskell and Lisp
- Neither of these definitions were accepted as the satisfactory definition for computability

Turing Machine

- Alan Turing
  - Thought about what humans do when they solve problems
  - We read some symbols on a piece of paper, think, and then make a decision
- Turing Machine mimics this behavior
- Consists of a tape of infinite length and a tape head
  - Tape is divided into cells that contain a symbol
  - Tape head reads a symbol from a cell and then makes a decision to either write a new symbol onto the cell or move

Turing Machine

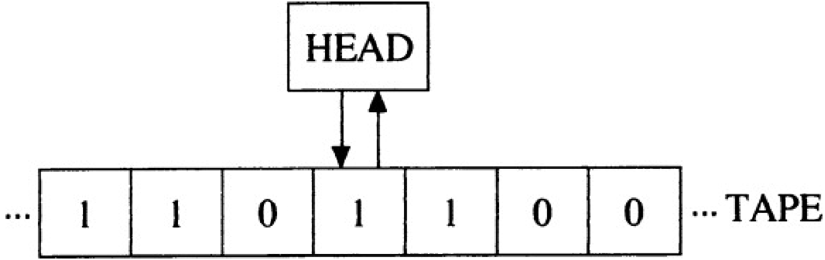


Figure: a visual representation of a Turing Machine.

Turing Machine

- Why were Turing Machines chosen as the model for computation?

$\Rightarrow$ Based on human behavior; intuitive to use

$\Rightarrow$ Mechanical aspect; visualize step-by-step process

Church-Turing Thesis

- Did we finally capture the full notion of computability?
  - We can never prove that we have done so
  - Requires an equivalence between a formal definition and an intuitive understanding
  - But, it turns out that partial recursive functions, Lambda functions, and Turing Machines are all equivalent!

Church-Turing Thesis

A function is computable iff it is Turing-computable, i.e., there is an equivalent Turing Machine.

Aside: Enumerating Turing Machines

- There is a computable bijection between the set of Turing Machines and $\mathbb{N}$
  - We can "translate" between Turing Machines and the natural numbers
  - Translation is done in a computable manner in both directions
  - The encoding of a Turing Machine is known as its **index**
- Notation: φ_e
  - Turing Machine with index e
  - Or, the eth Turing Machine

Computable Functions

Recursion Theorem

Recursion Theorem

Let f be a total computable function. Then there is an index n such that $\varphi_n = \varphi_{f(n)}$.

We will use the Recursion Theorem to prove Rice's Theorem.

Index Sets

Definition

Let $A \subseteq \mathbb{N}$. For any x and y, if we have that $x \in A$ and $\varphi_x = \varphi_y$ implies that $y \in A$, then A is an **index set**.

Index Sets

Examples:

- Fin $= \{e \mid \text{dom } \varphi_e < \infty\}$
  - Computable functions with finite domains
- Tot $= \{e \mid \text{dom } \varphi_e = \mathbb{N}\}$
  - Total computable functions

Basically "cherry-picking" computable functions based on what they do (semantic information)

Rice's Theorem shows us that we cannot do this "cherry-picking" in a computable manner

Rice's Theorem

Rice's Theorem

Suppose A is a nontrivial index set, i.e.,

$$\varnothing \subsetneq A \subsetneq \mathbb{N}.$$

Then χ_A is noncomputable.

Recall that for a set A, its characteristic function is defined as:

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Proving Rice's Theorem

- We will prove by contradiction.
- Suppose A is a nontrivial index set and that χ_A is computable.
- Since $\varnothing \subsetneq A \subsetneq \mathbb{N}$, then we can fix $a \in A$, $b \notin A$.
- Define f as follows:

$$f(x) = \begin{cases} a & \chi_A(x) = 0 \\ b & \chi_A(x) = 1 \end{cases}$$

Proving Rice's Theorem

- Since we assumed that χ_A is computable, then f is also computable.
- Additionally, since χ_A is a characteristic function, then it must be total. Thus, f is also total.
- Therefore, f is a total computable function.
- By the Recursion Theorem, there is some index n such that $\varphi_n = \varphi_{f(n)}$.

Proving Rice's Theorem

- We have two cases:
  1. If $n \in A$, then $f(n) = b \notin A$. But this contradicts our definition of an index set because if $n \in A$ and $\varphi_n = \varphi_b$, then we should have $b \in A$.
  2. If $n \notin A$, then $f(n) = a \in A$. Again, we have a contradiction of our definition of index sets because if $a \in A$ and $\varphi_a = \varphi_n$, then we should have $n \in A$.
- In both cases, we have a contradiction.
- Therefore, our assumption was incorrect and χ_A must be a noncomputable function. ■

Computable and Computably Enumerable Sets

Computable Sets

Definition

A set is **computable** if its characteristic function is computable.

Examples:

- $A = \{10, 15, 19, 5\}$
- $B = \{n \in \mathbb{N} \mid \exists k \in \mathbb{N} \text{ s.t. } n = 2k\} = \{0, 2, 4, 6, \ldots\}$

Computably Enumerable Sets

Definition

A set is **computably enumerable** if there is a computable procedure that outputs all the elements of the set, allowing repeats and does not have to respect an order.

Think of the procedure as an infinitely-printing printer, and the set as its receipt

Example: The Halting Set

- The set $K = \{e \mid \varphi_e(e) \downarrow\}$ is known as the *halting set*
  - The set of computable functions that halt on its index
- K is noncomputable
- However, K is computably enumerable
  - Step 1: Run one step of $\varphi_0(0)$
  - Step 2: Run another step of $\varphi_0(0)$, and then run two steps of $\varphi_1(1)$
  - Step 3: Run another step of $\varphi_0(0)$ and $\varphi_1(1)$, and then run three steps of $\varphi_2(2)$
  - Step i: Run i steps of $\varphi_0(0)$ to $\varphi_{i-1}(i-1)$
  - If any of the computations converge at any point, output the index

  $\Rightarrow$ Dovetailing

Computable vs Computably Enumerable Sets

- Difference is in the waiting time
- Computable Sets
  - We can know whether or not an element is in the set within a finite amount of time
- Computably Enumerable Sets
  - Keep waiting until the element is enumerated
  - If the element is in the set, then it is guaranteed that it will be enumerated at a certain point because the procedure enumerates all elements of the set
  - If the element is not in the set, then we are just waiting for something that will never come

Turing Reductions

Oracle Turing Machine

- Oracle Turing Machine
  - Turing Machine hooked up to a black box, known as the *oracle*
  - The oracle knows information about a particular set, say A
  - During computation, the Turing Machine can ask the oracle if a number is in A
- Notation: φ_e^A
  - eth Turing Machine with oracle A

Turing Reductions

Definition

Let $A, B \subseteq \mathbb{N}$. If there is an index e such that $\varphi_e^B = \chi_A$, then A is **Turing-reducible** to B, denoted by $A \leq_T B$

- We are using answers from χ_B to calculate the answer for χ_A
- In other words, if we know how to solve B, then we can solve A
- We are *reducing* the problem from A to B

Turing Degrees

Turing Equivalence

Definition

Let $A, B \subseteq \mathbb{N}$. If $A \leq_T B$ and $B \leq_T A$, then A and B are **Turing equivalent**, denoted by $A \equiv_T B$.

Turing Degrees

- Turing Equivalence is an equivalence relation
- Thus, you can take the quotient of $\mathcal{P}(\mathbb{N})$ by $\equiv_T$
    - i.e., partition sets of natural numbers by Turing equivalence
- Each equivalence class ("slice") is known as a **Turing degree**

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