DPR Topic: Lie Groups/Lie Algebras

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Section Overview

Definition

A lie group is a smooth differentiable manifold that is also a group

All Lie groups discussed will be matrix Lie groups

What is a manifold

A manifold is a space that looks like a Euclidean space when zoomed in close enough

Example

Surface of a sphere: The surface of a sphere appears like Euclidean space on small enough scales

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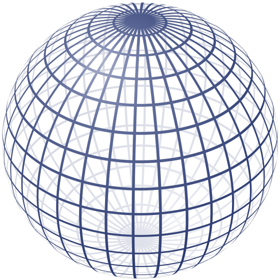

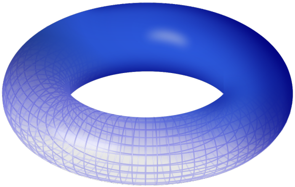

Definition

A Lie Algebra is

1. a finite dimensional real or complex vector space
2. endowed with a map $[\cdot,\cdot]$ such that
    1. $[\cdot,\cdot]$ is bilinear (linear with respect to both inputs)
    2. is "skew symmetric", is antisymmetric so $[X,Y] = -[Y,X]$
    3. satisfies the Jacobi identity
       $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

The map is also called the bracket or the commutator

Exponential of Matrix

Definition

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

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Properties

1. Converges for all matrices X
2. Easily computable if diagonalizable
3. Conserves the original definition of the exponential map

Logarithm of a Matrix

Definition

$$log(X) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(X - I)^m}{m}$$

Properties

1. Does not always converge
2. $log(e^X) = X$ when X is diagonizable and all of the eigenvalues are within the unit circle in the complex plane

Section Overview

1. Basic Definitions
   - Lie Groups
   - L Algebra
   - Exponential of a Matrix
2. Relationships Between Lie Algebras and Lie Groups
   - Correspondences Between Lie Algebras and Lie Groups
   - Formulas Relating Lie Algebras and Lie Groups

The Lie Algebra of a Matrix Lie Group

Definition

Given a Lie group, G, its Lie algebra is the set of all matrices such that $e^{tX} \in G$ for all real values of t

For this Lie algebra, we use [X,Y]=XY-YX as thie LIe bracket.

Theorem

Every LIe group homomorphism gives rise to a LIe algebra homomorphism

Theorem

If $\mathfrak{g}$ is any finite dimensional real Lie algebra, there exists a connected Lie subgroup G of $GL(n;\mathbb{C})$ whose lie algebra is isomorphic to $\mathfrak{g}$

Baker-Campbell-Hausdorff Formula

Theorem

For all $n \times n$ matrices X and Y with $\| X \|$ and $\| Y \|$ sufficiently small

$$log(e^X e^Y) = X + \int_0^1 g(e^{ad_X} e^{tad_Y})(Y)dt$$

where $g(z) = \dfrac{log(z)}{1 - \frac{1}{z}}$ *and* $ad_X(z) = [X, z]$

Homomorphisms Between Lie Algebras and Lie Groups

Theorem

Let G and H bet Lie matrix groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. and let $\phi : \mathfrak{g} \mapsto \mathfrak{h}$ be a Lie algebra homomorphism. If G is simply connected then there exists a unique Lie group homomorphism $\Phi : G \mapsto H$ such that $\Phi(e^X) = e^{\phi(X)} \forall X \in \mathfrak{g}$

For the program I mainly used Brian C Halls Lie Groups, Lie Algebras and Representations