Measure Theory and an Introduction to Fractals

Julian Ivaldi

Overview

- Measures and some examples
- Hutchinson’s theorem and its proof
- Applications of fractal geometry

Motivation

What is length?

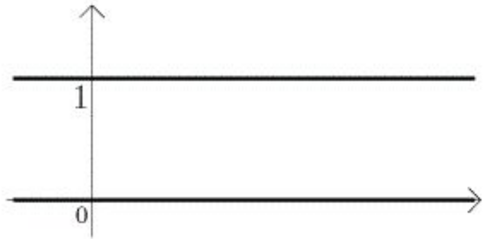
How do we integrate certain discontinuous functions?

The Dirichlet function

Fractal sets can be troublesome too.

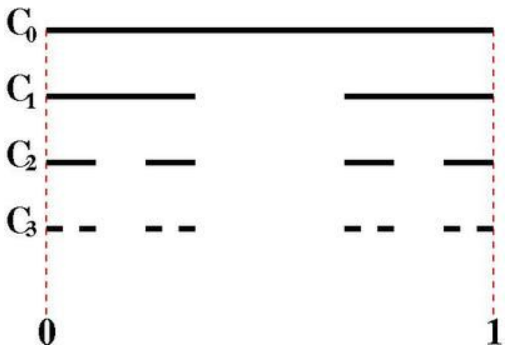
Cantor set

Koch curve

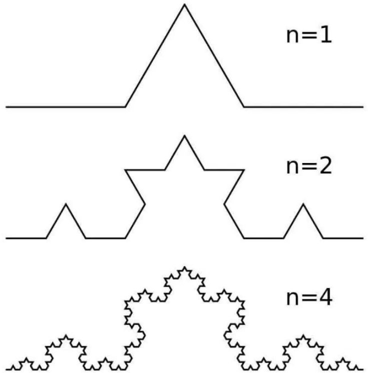


$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

The definition of the Dirichlet function, with a graph of the function above.



The first four iterations of the Cantor set construction.



Several steps in the construction of the Koch curve.

How is a Measure Defined?

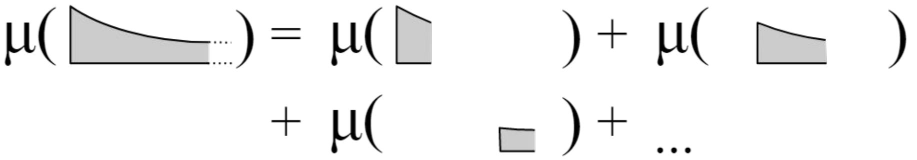
Measures are functions defined on σ-algebras.

$\mathscr{A} \subset 2^X$ is a σ-algebra of a set X if

1. $\emptyset, X \in \mathscr{A}$.
2. If a countable collection of sets $\{A_i\} \in \mathscr{A}$, then $\cup_{i=1}^{\infty} A_i \in \mathscr{A}$.

A function $\mu : \mathscr{A} \to [0, \infty]$ is a measure if

1. $\mu(\emptyset) = 0$
2. μ is countably additive. That is, $\mu(A \cup B) = \mu(A) + \mu(B)$



A visualization of the additivity of a measure.

Lebesgue Measure

Finds area of sets in R^n by adding half-open rectangles

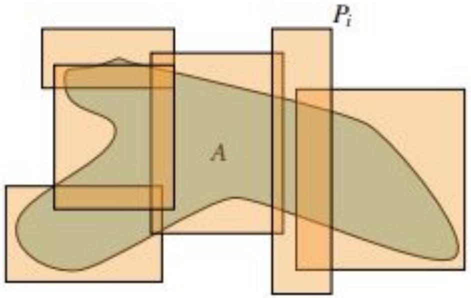
First, we define the area of rectangles

$$P = \{(x_1, ..., x_n) : \ x_i \in [a_i, b_i)\}$$

$$|P| = \prod_{i=1}^{n} (b_i - a_i)$$

Then, we define the Lebesgue measure as

$$\mathscr{L}(A) = \inf \left\{ \sum_{i=1}^{\infty} |P_i| : \{P_i\} \text{ is a collection of half-open rectangles s.t. } A \subset \cup_{i=1}^{\infty} P_i \right\}$$



An example of a collection of half open rectangles that covers a set A.

Hausdorff measure

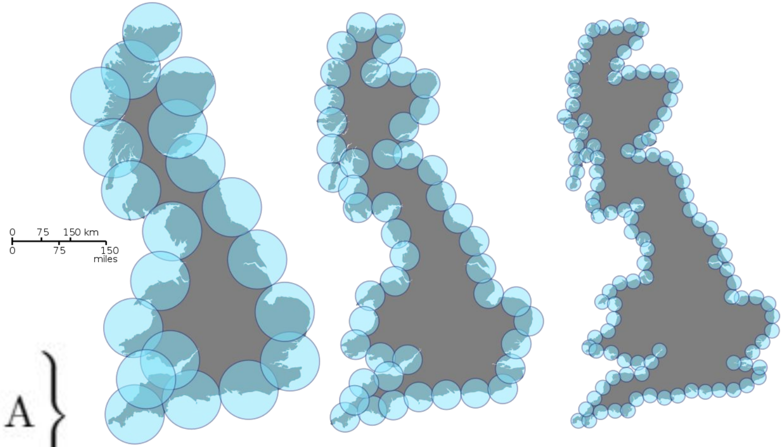
Construction starts with δ-coverings.

A collection of sets $\{E_i\}$ is a δ-covering of a set A if

1. $\text{diam}(E_i) < \delta \quad i \in \mathbb{N}$
2. $A \subset \cup_{i=1}^{\infty} E_i$

From this, we define a measure H^s_δ as follows.

$$\mathscr{H}^s_\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(E_i)^s : \{E_i\} \text{ is a } \delta\text{-covering of A} \right\}$$



An image demonstrating different δ-coverings of the coast of Great Britain.

The s-Dimensional Hausdorff Measure

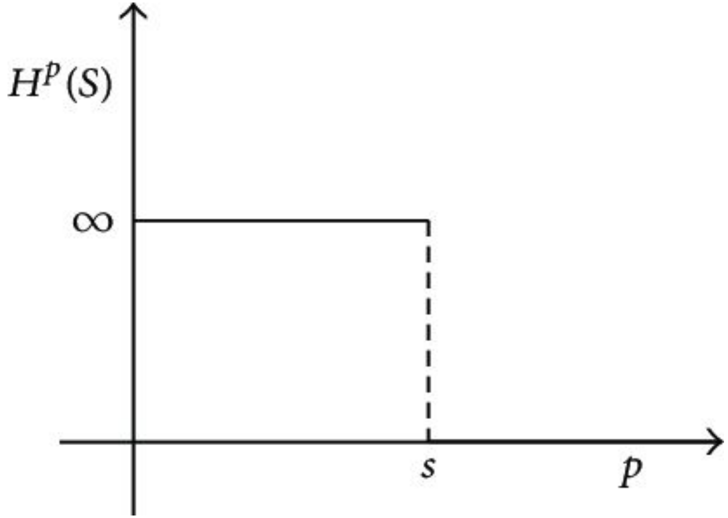
We define the s-dimensional Hausdorff measure as such

$$\mathscr{H}^s = \sup_{\delta>0} \mathscr{H}^s_\delta = \lim_{\delta\to 0} \mathscr{H}^s_\delta$$

The Hausdorff measure has a useful property

For some s, t where $0 \leq s < t < \infty$ and a set $A \subset X$,

1. If $\mathscr{H}^s(A) < \infty$, then $\mathscr{H}^t(A) = 0$
2. If $\mathscr{H}^t(A) > 0$, then $\mathscr{H}^s(A) = \infty$



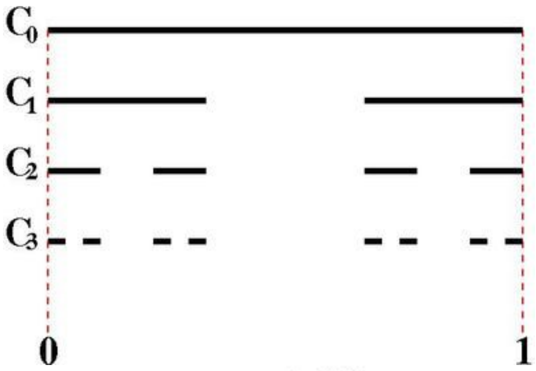
A graph of the p-dimensional Hausdorff measure with respect to p for some set. See that at all p>s, the measure is 0, and at p<s, it is infinity.

The Hausdorff Dimension

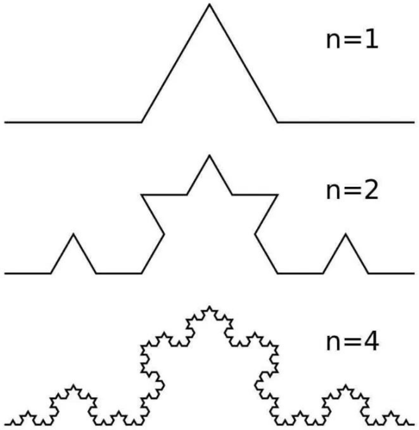
The Hausdorff dimension is the value where the Hausdorff measure changes.

$$\dim_{\mathscr{H}}(A) = \sup(s \geq 0, \mathscr{H}^s(A) = \infty) = \inf(t \geq 0, \mathscr{H}^t(A) = 0)$$

We can reconsider our fractal examples from before:



$$\dim_{\mathscr{H}}(C) = \frac{\ln(2)}{\ln(3)} \approx 0.631$$



$$\dim_{\mathscr{H}}(K) = \frac{\ln(4)}{\ln(3)} \approx 1.262$$

Hutchinson's Theorem

I'll first state Hutchinson's theorem, then discuss the relevant definitions and concepts. After which, the proof of the theorem is actually rather brief.

Hutchinson's theorem states:

Given a complete metric space X and a collection of contractions $\{f_i\}_{1 \leq i \leq n}$ where $f_i : X \to X$, there exists a unique compact set $K \subset X$ that satisfies the following,

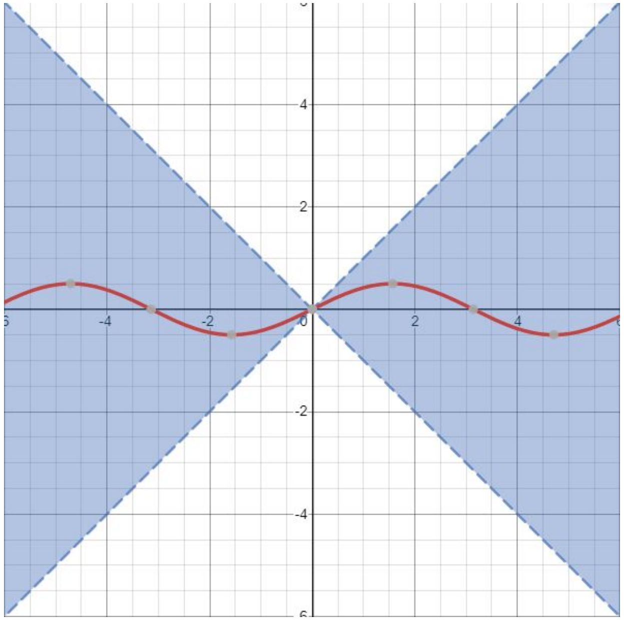
$$K = \bigcup_{i=1}^{n} f_i(K)$$

Contractions

Contractions bring points in X closer together in the image of X. Specifically,

$f : X \to X$ is a contraction on X if for all $x, y \in X$,
$d(f(x), f(y)) \leq r d(x, y)$ for some contraction ratio $r < 1$

Notice that this is the definition of a Lipschitz continuous mapping with the added requirement that the Lipschitz constant be less than 1.



f(x)=sin(x)/2 is an easily proven example of a contraction on the real numbers.

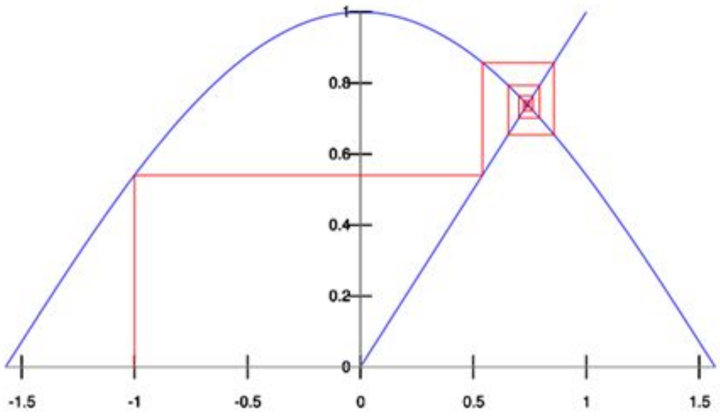
Fixed Point Theorem

The fixed point theorem is as follows.

Given a contraction f on a complete metric space X, there exists a unique point $x \in X$ such that $f(x) = x$

In the proof of the theorem, the completeness of X proves the existence of x by the construction of a cauchy sequence.

In fact, starting at any point in X, repeated applications of f will approach x, hence the uniqueness.



The iterative process described is shown by this cobweb diagram.

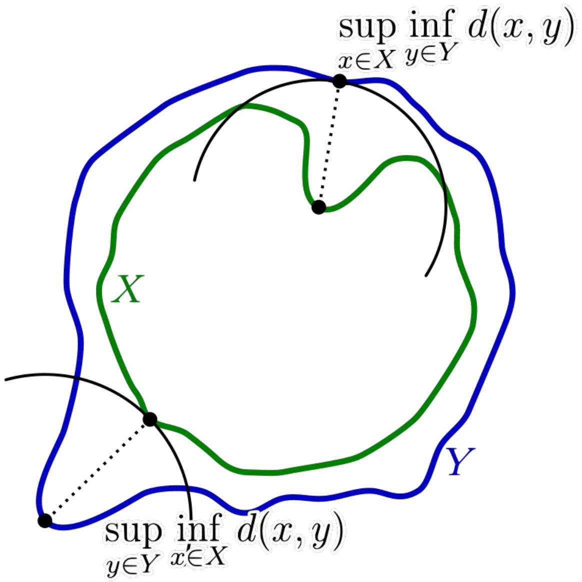
Hausdorff distance

The Hausdorff distance is a measurement of distance between two sets. It’s defined on all non-empty subsets of X

Given a metric space X and $A, B \subset X$, we define,

$$h(A,B) = \max(\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A))$$

The set of all non-empty compact subsets of X, called H(X), becomes a metric space when endowed with the Hausdorff distance.



The Hausdorff distance between two sets X and Y, showing the two different supremums in the definition.

Properties of (H(X),h)

The Hausdorff metric has useful properties similar to those in the original space.

For finite collections of sets $\{A_i\}$ and $\{B_i\}$ in X,

$$h\left(\bigcup_{i=1} A_i, \bigcup_{i=1} B_i\right) \leq \max_{1 \leq i \leq n} (h(A_i, B_i))$$

For a Lipschitz continuous function $f : X \to X$ with Lipschitz constant k and $A, B \subset X$,

$$h(f(A), f(B)) \leq kh(A, B)$$

If X is a complete metric space, then $(H(X), h)$ is also complete.

Return to the Theorem

Given a complete metric space X and a collection of contractions $\{f_i\}_{1 \leq i \leq n}$ where $f_i : X \to X$, there exists a unique compact set $K \subset X$ that satisfies the following,

$$K = \bigcup_{i=1}^{n} f_i(K)$$

Looks a lot like the fixed point theorem!

To prove, we'll make a contraction on the compact metric space H(X), then apply fixed point theorem to find a unique invariant set.

Proof of Hutchinson's Theorem

Recalling the properties of Hausdorff metric, see the following:

Let k_i be the contraction constant of f_i. Let $F(A) = \bigcup_{i=1}^{n} f_i(A)$. Then,

$$\begin{aligned} h(F(A), F(B)) &= h(\bigcup_{i=1}^{n} f_i(A), \bigcup_{i=1}^{n} f_i(B)) \\ &\leq \max_i h(f_i(A), f_i(B)) \\ &\leq \max_i k_i h(A, B) \end{aligned}$$

Since F(A) is a contraction on a complete metric space (H(X),h), by the fixed point theorem there exists a point K (a subset of X) in H(X) such that K = F(K). By applying F recursively to an arbitrary set, we can approach K to an arbitrary distance.

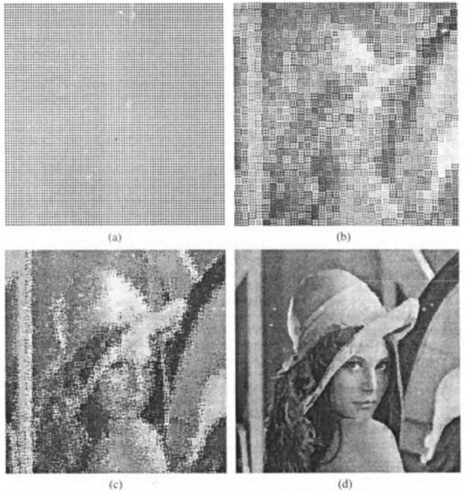
Applications

Image compression

Find rules for arbitrary image, work backwards. Get contractions from image.

Biological processes

Plant growth, cell division, surface area maximization.



A reconstruction of an image stored using fractal compression. Pictured are 0, 1, 2, and 10 iterations.

The Barnsley fern, a fractal representation of a black spleenwort fern.

Image sources in order of appearance

All LaTeX written in Overleaf by me

https://mathblab.tumblr.com/post/144378845420/dirichlet-function-notice-the-graph-looks-like-2

https://tasks.illustrativemathematics.org/content-standards/tasks/929

https://fractal.institute/old/koch-curve/

https://en.wikipedia.org/wiki/Measure_(mathematics)

Measure and Integration, Vasilis Chousionis

https://en.wikipedia.org/wiki/Hausdorff_dimension

https://www.researchgate.net/publication/264745668_Fractal_Analysis_of_Laplacian_Pyramidal_Filters_Applied_to_Segmentation_of_Soil_Images

https://sites.google.com/site/procesosnumericos0241/fixed-point-method

https://steelpangolin.wordpress.com/2014/07/09/a-review-of-fractal-image-compression-and-related-algorithms/

https://en.wikipedia.org/wiki/Barnsley_fern