Fuzzy Logic

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Introduction

In Mathematics, the **set** is a rudimentary concept. From sets, or **collections of objects**, we can make statements about the objects within them, connections between sets, and even create mappings from one set to another.

Introduction

Objects are related to classical sets by a clear, crisp sense of membership. For some object x, there are only two possibilities:

$$x \in A$$

or

$$x \notin A.$$

When dealing with the abstract, classical sets serve their purpose. But in the real-world (as we all know too well) there is uncertainty and ambiguity.

Fuzzy Sets

Fuzzy sets are actually generalizations of classical sets, which are defined by the characteristic function $\chi_A(x)$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Fuzzy Sets

A fuzzy set is a class (collection of sets or objects) with a continuum of membership grades.

Definition

A fuzzy set A (subset of X) is defined as a mapping

$$A : X \to [0, 1]$$

where $A(x)$ is the membership degree of x to the fuzzy set A. We denote by $\mathcal{F}$ the collection of all fuzzy subsets of X

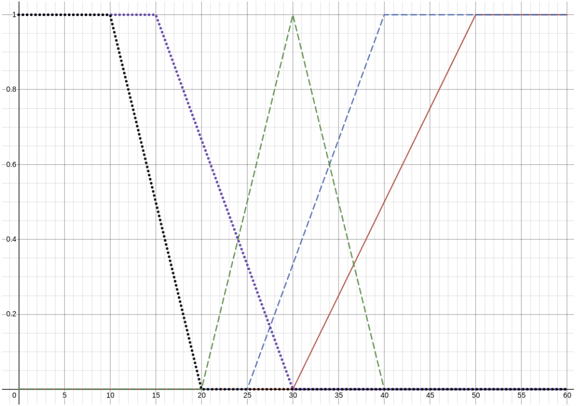
Fuzzy Sets

Fuzzy sets allow for a robust definition of sets or categories that can handle the obscurity that comes with language and physical systems.

Ex: To what extent are certain car speeds "slow", "average", "fast", etc.?

Fuzzy Sets

$$\text{"very slow" (black): } A(x) = \begin{cases} 1 & \text{if } 0 \le x < 10 \\ \frac{20-x}{10} & \text{if } 10 \le x < 20 \\ 0 & \text{if } 20 \le x \le 60 \end{cases}$$



Connectives

Definition (Intersection)

Let $A, B \in \mathcal{F}(X)$. The **intersection** of A and B is the fuzzy set C with

$$C(x) = min\{A(x), B(x)\} = A(x) \wedge B(x), \forall x \in X.$$

We denote $C = A \wedge B$.

Definition (Union)

Let $A, B \in \mathcal{F}(X)$. The **union** of A and B is the fuzzy set C with

$$C(x) = max\{A(x), B(x)\} = A(x) \vee B(x), \forall x \in X.$$

We denote $C = A \vee B$.

Definition (Complementation)

Let $A \in \mathcal{F}(X)$ be a fuzzy set. The **complement** of A is the fuzzy set where

$$B(x) = 1 - A(x), \forall x \in X$$

Fuzzy Relations

Definition (Classical Relation)

A subset $R \subseteq X \times Y$ where X and Y are classical sets, is a classical relation.

Similar to classical sets, a classical relation can be characterized by a function $R : X \times Y \to \{0, 1\}$,

$$R(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R \\ 0 & \text{if } (x, y) \notin R \end{cases}$$

Definition (Fuzzy Relation)

Let X, Y be two classical sets. A mapping $R : X \times Y \to [0, 1]$ is called a **fuzzy relation**. *The number $R(x, y) \in [0, 1]$ can be interpreted as the degree of relationship between x and y.*

Fuzzy Relations

If X and Y are finite sets such that $X = \{x_1, x_2, ...x_n\}$, $Y = \{y_1, y_2, ...y_n\}$ a fuzzy relation between the two sets can be represented as the following matrix:

$$R = \begin{pmatrix} R(x_1, y_1) & R(x_1, y_2) & ... & R(x_1, y_n) \\ R(x_2, y_1) & R(x_2, y_2) & ... & R(x_2, y_n) \\ ... & ... & ... & ... \\ R(x_m, y_1) & R(x_m, y_2) & ... & R(x_m, y_n) \end{pmatrix}$$

Max-Min Compositions

Definition

Let $R \in \mathcal{F}(X \times Y)$ and $S \in \mathcal{F}(Y \times Z)$ be fuzzy relations. Then $R \circ S(x,z) \in \mathcal{F}(X \times Z)$, defined as

$$R \circ S(x,z) = \bigvee_{y \in Y} R(x,y) \wedge S(y,z),$$

is the max-min composition *of the fuzzy relations R and S.*

Max-Min Compositions

Max-Min Compositions for Finite Sets

Let $X = \{x_1, ..., x_n\}$, $Y = \{y_1, ..., y_m\}$, $Z = \{z_1, ..., z_p\}$ be finite sets. If $R = (r_{ij})_{i=1,\ldots,n, j=1,\ldots,m} \in \mathcal{F}(X \times Y)$ and $S = (s_{jk})_{j=1,\ldots,n, k=1,\ldots p} \in \mathcal{F}(X \times Y)$ are discrete fuzzy relations then the composition $T = (t_{ik})_{j=1,\ldots,n, k=1,\ldots p} = R \circ S \in \mathcal{F}(X \times Z)$ is given by

$$t_{ik} = \bigvee_{j=1}^{m} r_{ij} \wedge s_{jk},$$

$i = 1, ..., n, k = 1, ..., p$

Example: If $R = \begin{pmatrix} 0.3 & 0.7 & 0.2 \\ 1 & 0 & 0.9 \end{pmatrix}$ and $S = \begin{pmatrix} 0.8 & 0.3 \\ 0.1 & 0 \\ 0.5 & 0.6 \end{pmatrix}$ then

$R \circ S = \begin{pmatrix} 0.3 & 0.3 \\ 0.8 & 0.6 \end{pmatrix}$

Medical Applications

Fuzzy logic systems have been used for modelling diagnosis in medicine.

The max-min composition allows for a mapping from a set of patients to a set of symptoms, then from symptoms to a set of diagnoses.[Samuel and Balamurugan, 2012]

Negation

Definition

A function $N : [0,1] \to [0,1]$ is called a negation if $N(0) = 1, N(1) = 0$ and N is non-increasing ($x < y \implies N(x) > N(y)$). A negation is called a **strict negation** if it is strictly decreasing ($x < y \implies N(x) > N(y)$) and continuous. A strict negation is said to be a **strong negation** if $N(N(x)) = x$.

Triangular Norms and Conorms

Triangular norms and conorms are generalizations of the basic connectives of fuzzy sets.

Definition

Let $T, S : [0,1]^2 \rightarrow [0,1]$*. Consider the following properties:*

T_1 :T(x,1) = x (*identity*)
S_1 :S(x,0) = x
T_2 :T(x,y) = T(y,x) (*commutativity*)
S_2 :S(x,y) = S(y,x)
T_3 :T(x,T(y,z))=T(T(x,y),z) (*associativity*)
S_3 :S(x,S(y,z))=S(S(x,y),z)
T_4 : *If* x$\leq u$ and $y \leq v$ then $T(x,y) \leq T(u,v)$ (monotonicity)
S_4 : *If* x$\leq u$ and $y \leq v$ then $S(x,y) \leq S(u,v)$

A **triangular norm** (t-norm) is a function $T : [0,1]^2 \rightarrow [0,1]$ that satisfies $T_1 - T_4$

A **triangular conorm** (t-conorm) is a function $S : [0,1]^2 \rightarrow [0,1]$ that satisfies $S_1 - S_4$

Ex: Gödel, Gougen t-norm and t-conorm

Gödel t-norm, t-conorm, standard negation
$x \wedge y = min\{x, y\}$
$x \vee y = max\{x, y\}$
$N(x) = 1 - x$

T_1 :$\min\{x, 1\} = x$

S_1 :$\max\{x, 0\} = x$

T_2 :$\min\{x, y\} = min\{y, x\}$

S_2 :$\max\{x, y\} = max\{y, x\}$

T_3 :$\min\{x, min\{y, z\}\} = min\{min\{x, y\}, z\}$

S_3 :$\max\{x, max\{y, z\}\} = max\{max\{x, y\}, z\}$

T_4 : If x$\leq u$ and $y \leq v$ then $min\{x, y\} \leq min\{u, v\}$

S_4 : If x$\leq u$ and $y \leq v$ then $max\{x, y\} \leq max\{u, v\}$

Gödel, Gougen t-norm and t-conorm

Gougen t-norm, t-conorm, standard negation

$xT_G y = x \cdot y$

$xS_G y = x + y - xy$

$N(x) = 1 - x$

$T_1 : x \cdot 1 = x$

$S_1 : x + 0 - 0 = x$

$T_2 : x \cdot y = y \cdot x$

$S_2 : x + y - xy = y + x - yx$

$T_3 : x \cdot (yz) = (xy) \cdot z$

$S_3 : x + (y + z - yz) - x(y + z - yz) = (x + y - xy) + z - z(x + y - xy)$

T_4 : *If* $x \leq u$ and $y \leq v$ then $x \cdot y \leq u \cdot v$

S_4 : *If* $x \leq u$ and $y \leq v$ then $x + y - xy \leq u + v - uv$

DeMorgan Triplets

A triplet (S, T, N) is called a **De Morgan triplet** if T is a t-norm, S is a t-conorm, N is a strong negation, and if they fulfill De Morgan's law

$$S(x, y) = N(T(N(x), N(y)))$$

Example 2.15. The minimum, maximum, and standard negation

$$x \wedge y = min\{x, y\}$$
$$x \vee y = max\{x, y\}$$
$$N(x) = 1 - x$$

form a De Morgan triplet

Multiple Columns

Lukasiewicz t-norm, t-conorm, standard negation

$(x + y - 1) \vee 0$

$(x + y) \wedge 1$

$N(x) = 1 - x$

$$N(T(N(x), N(y))) = S(x, y) \leftrightarrow (x + y) \wedge 1 =$$
$$((1 - x) + (1 - y) - 1) \vee 0 \leftrightarrow min\,\{x + y, 1\} = max\,\{1 - x - y, 0\}$$

Case 1: $1 - x - y < 0 \implies 1 < x + y$

Case 2: $1 - x - y > 0 \implies 1 > x + y$

References

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