An Introduction to Fractal Analysis

Julian Ivaldi

April 29, 2021

Contents

σ-Algebras and Measures

A measure on a set is a notion of area or weight of certain subsets of that set. These subsets must be a part of a σ-algebra, which is the structure required to define measures. There exist weaker requirements such as algebras or semi-algebras that alone do not suffice to define measures but are still useful.
A measure μ on a σ-algebra $\mathcal{A}$ is defined as a function $\mu : \mathcal{A} \to [0, \infty]$ that satisfies

1. $\mu(\emptyset) = 0$
2. For $A \subset B$, $\mu(A) \leq \mu(B)$ (**Monotonicity**)
3. For a countable collection of sets $\{A_i\}$ in $\mathcal{A}$, $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$ (**Additivity**)

Lebesgue Measure, Hausdorff Measure, and More

$$\mathcal{L}^{*n}(A) = \inf\{\sum_i |P_i|, A \subset \bigcup_i P_i, P_i \text{ half open rectangles}\}$$

$$\mathcal{H}^\alpha_\delta(A) = \inf\{\sum_i |E_i|^\alpha, A \subset \bigcup_i E_i, |E_i| < \delta\}$$

$$\mathcal{H}^\alpha(A) = \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(A)$$

$$\mathcal{H}^\alpha_\infty(A) = \inf\{\sum_i |E_i|^\alpha, A \subset \bigcup_i E_i\}$$

$$\mathcal{H}^\phi_\infty(A) = \inf\{\sum_i \phi(|E_i|), A \subset \bigcup_i E_i\}$$

Fractal Dimension

We define the Hausdorff dimension of a set A to be

$$\mathsf{dim}(A) = \sup(\{\alpha : \mathcal{H}^{\alpha}(A) = \infty\}) = \inf(\{\alpha : \mathcal{H}^{\alpha}(A) = 0\})$$

Other ways to define dimension include Minkowski dimension and packing dimension. These other dimensions may bound Hausdorff dimension.

Hutchinson's Theorem

Hutchinson's theorem states that given a complete metric space (X, d) and a family of contractions $\{f_i\}_{i=1}^{\ell}$ on X,

1. There exists a unique non-empty compact set K such that

$$K = \bigcup_{i=1}^{\ell} f_i(K)$$

2. For any probability vector $\mathbf{p} = (p_1, ..., p_\ell)$ there exists a unique probability measure $\mu_{\mathbf{p}}$ on the attractor K such that

$$\mu_{\mathbf{p}} = \sum_{i=1}^{\ell} p_i \mu_{\mathbf{p}} f_i^{-1}$$

If $p_i > 0$ for all i, then $\text{supp}(\mu_{\mathbf{p}}) = K$.

Banach Fixed-Point Theorem

The Banach fixed-point theorem states that for a complete metric space X and a contraction f, there exists a unique fixed point $z \in X$ such that $f(z) = z$.

We can make use of this theorem by constructing an appropriate metric space and contraction which prove the existence and uniqueness of a fixed point in that space with desired properties.

Fixed Point Theorem for the First Claim

For the first statement of Hutchinson's theorem, we consider the space $\mathsf{Cpt}(X)$ defined as all compact subsets of X. Since X is complete, $\mathsf{Cpt}(X)$ endowed with the Hausdorff metric d_H is a complete metric space (Blaschke's selection theorem).

It can be shown that the function $F = \bigcup_{i=1}^{\ell} f_i$ is a contraction on $(\mathsf{Cpt}(X), d_H)$ and as such, we can apply the fixed-point theorem to obtain a point $K \in \mathsf{Cpt}(X)$ such that $K = F(K) = \bigcup_{i=1}^{\ell} f_i(K)$.

Proof of Second Claim: Metric Space and Self-Map

We follow a similar process to prove the second claim, this time using the space $P(K)$ of Borel probability measures on K. This is a compact metric space when given the dual Lipschitz metric,

$$L(\mu, \nu) = \sup_{\mathsf{Lip}(g) \leq 1} \left| \int g d\mu - \int g d\nu \right|$$

We define a self-map $F_{\mathbf{p}}$ on $P(K)$ as follows

$$F_{\mathbf{p}}(\nu) = \sum_{i=1}^{\ell} p_i \nu f_i^{-1}$$

It remains to show that this is a contraction on $(P(K), L(\mu, \nu))$ and to prove the final note about $\mathsf{supp}(\mu)$.

Proof of Second Claim: $F_{\mathbf{p}}$ is a Contraction

We first note that for some function $g : K \to \mathbb{R}$ with Lip$(g) \leq 1$, the Lipschitz norm Lip$(\sum_{i=1}^{\ell} p_i g f_i) \leq r_{\max}$. Now,

$$L(F_{\mathbf{p}}(\mu), F_{\mathbf{p}}(\nu)) = \sup_{\text{Lip}(g) \leq 1} \left| \int g dF_{\mathbf{p}}(\mu) - \int g dF_{\mathbf{p}}(\nu) \right|$$

$$\left| \int g dF_{\mathbf{p}}(\mu) - \int g dF_{\mathbf{p}}(\nu) \right| = \left| \int \sum_{i=1}^{\ell} p_i g f_i d\mu - \int \sum_{i=1}^{\ell} p_i g f_i d\nu \right|$$

$$\leq \text{Lip} \left(\sum_{i=1}^{\ell} p_i g f_i \right) L(\mu, \nu) \leq r_{\max} L(\mu, \nu)$$

Thus, we see that $F_{\mathbf{p}}$ is a contraction. We can apply the fixed point theorem to obtain $\mu_{\mathbf{p}}$ that satisfies $\mu_{\mathbf{p}} = \sum_{i=1}^{\ell} p_i \mu_{\mathbf{p}} f_i^{-1}$.

Support of $\mu_{\mathbf{p}}$ is K

If $p_i > 0$ for all i, then given a probability measure $\nu \in P(X)$ with bounded support that satisfies $\nu = \sum_{i=1}^{\ell} p_i \nu f_i^{-1}$, we see that $\text{supp}(\nu)$ will satisfy

$$\text{supp}(\nu) = \bigcup_{i=1}^{\ell} f_i(\text{supp}(\nu))$$

Since the support of a measure is always closed, we find that $\text{supp}(\mu_{\mathbf{p}}) = K$ from the uniqueness of K.

The Mass Distribution Principle

The Mass Distribution Principle states that if a set E supports a Borel measure μ where

$$\mu(B(x, r)) \leq Cr^{\alpha}$$

for all balls $B(x, r)$ and some constant $0 < C < \infty$, then $\mathcal{H}^{\alpha}(E) \geq \frac{1}{C}\mu(E)$ and thus $\dim(E) \geq \alpha$.

Proof of MDP

Consider any cover $\{U_i\}$ of E. We choose $\{r_i\}$ such that $r_i > |U_i|$ and $\{x_i\}$ where $x_i \in U_i$. Then, we recall the assumption to state,

$$\mu(U_i) \leq \mu(B(x_i, r_i)) \leq C r_i^\alpha$$

We let r_i approach $|U_i|$ to conclude $\mu(U_i) \leq C|U_i|^\alpha$

$$\frac{1}{C}\mu(E) \leq \sum_i \frac{\mu(U_i)}{C} \leq \sum_i |U_i|^\alpha$$

Since $\{U_i\}$ was arbitrary, we conclude $\mathcal{H}^\alpha(E) \geq \mathcal{H}^\alpha_\infty(E) \geq \frac{1}{C}\mu(E)$ and thus that $\dim(E) \geq \alpha$

Frostman's Lemma

Frostman's Lemma states that for a gauge function ϕ and a compact set $K \subset \mathbb{R}^d$ with Hausdorff content $\mathcal{H}^{\phi}_{\infty}(K) > 0$, there exists a Borel measure μ on K such that $\mu(K) \geq \mathcal{H}^{\phi}_{\infty}(K)$ and for all balls B, $\mu(B) \leq C_d \phi(|B|)$

Trees, Flow, and Conductance

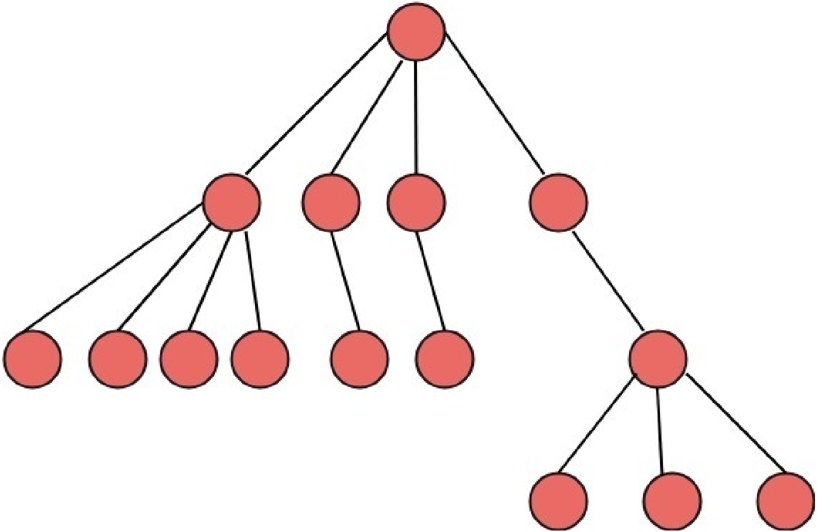
A **rooted tree** Γ is a collection of vertices and edges starting at a specific root vertex where there exists exactly one path through edges between any two vertices.
We denote the root vertex σ_0 and for a vertex σ, the depth from the root $|\sigma|$, and the adjacent vertex closer to the root σ'.
To each edge $\sigma'\sigma$, we assign a positive **conductance** $C(\sigma'\sigma)$.
We define a **flow** as a non-negative function f of edges such that

$$f(\sigma'\sigma) = \sum_{\tau'=\sigma} f(\sigma\tau)$$

A **legal flow** is one where $f(\sigma'\sigma) \leq C(\sigma'\sigma)$ for all σ.
The norm of a flow is defined as

$$||f|| = \sum_{|\sigma|=1} f(\sigma_0\sigma)$$

Example of a Tree



Cut-Sets and Minimal Cut-Sets

A **cut-set** is a set of edges Π that intersects all paths from the root. A **minimal cut-set** is one which has no proper subsets that are also cut-sets. Cut-sets have important properties. If we consider a flow f, then

$$||f|| \leq \sum_{e \in \Pi} f(e)$$

Equality holds when Π is a minimal cut-set. For legal flows,

$$||f|| \leq \sum_{e \in \Pi} f(e) \leq \sum_{e \in \Pi} C(e) := C(\Pi)$$

Max-Flow Min-Cut Theorem

The previous slide implies,

$$\max_{\text{legal flows}} ||f|| \leq \min_{\text{cut sets}} C(\Pi)$$

The max-flow min-cut theorem claims that equality holds for both finite and infinite trees, and most importantly that there exists a flow that attains said maximum value.

Return to Frostman's Lemma

To apply our knowledge of trees, we must construct an appropriate tree based on our assumptions.
Fix some integer $b > 1$ and construct the b-adic tree Γ corresponding to K. Vertices of depth n correspond to b-adic cubes of generation n that intersect K. Thus, all vertices are guaranteed to have a parent. We define conductance on Γ as

$$C(\sigma'\sigma) = \phi(\sqrt{d}b^{-n})$$

The max-flow min-cut theorem guarantees a maximal flow f for this conductance.

$\tilde{\mu}$ and extension to μ

We first construct a premeasure $\tilde{\mu}$ defined as such,

$$\tilde{\mu}(\{\text{all paths through } \sigma'\sigma\}) = f(\sigma'\sigma)$$

If we let S denote the collection of all sets of the form $\{\text{all paths through } \sigma'\sigma\}$ and $\emptyset$, then S is a semi-algebra. $\tilde{\mu}$ is additive by the conservation of flow, so it is a premeasure on S. By the extension theorem for semi-algebras, we can extend $\tilde{\mu}$ to the σ-algebra generated by S. Thus, we have constructed a measure μ which satisfies that $\mu(I_\sigma) = f(\sigma'\sigma)$. It remains to show that it has the desired properties.

$\mu(B) \leq C_d \phi(|B|)$

This property follows from the fact that any cube J can be covered by C_d smaller b-adic cubes, and by the increasing properties of ϕ,

$$\mu(J) \leq \sum_{i=1}^{C_d} \mu(I_\sigma) \leq C_d \phi(|J|)$$

$\mu(K) \geq \mathcal{H}^{\phi}_{\infty}(K)$

First, we recall the tree Γ corresponding to K. We note that any b-adic cover of K corresponds to a cut-set of Γ, so

$$\inf_{\Pi} C(\Pi) = \inf_{\Pi} \sum_{e \in \Pi} \phi(\sqrt{d} b^{-|e|}) \geq \tilde{\mathcal{H}}^{\phi}_{\infty}(K) \geq \mathcal{H}^{\phi}_{\infty}(K)$$

Thus, applying the max-flow min-cut theorem,

$$\mu(K) = ||f|| = \inf_{\Pi} C(\Pi) \geq \mathcal{H}^{\phi}_{\infty}(K)$$

Sources

Bishop, Peres, Fractals in Probability and Analysis

Chousionis, Measure Theory

Aldridge, Lecture 6, Constructing measures III: Caratheodory's extension theorem