Probability Prelim Exam for Actuarial Students
January 2021

Instructions

(a). The exam is closed book and closed notes.

(b). Answers must be justified whenever possible in order to earn full credit.

(c). Points will be deducted for mathematically incorrect statements.

1. (10 points) Let $\Omega = 1, 2, 3, 4$ and let $\mathscr{J} = \{\{1\}, \{2\}\}$.
   (a) (5 points) Describe explicitly the minimum semialgebra that contains $\mathscr{J}$.
   (b) (5 points) Describe explicitly the σ-algebra generated by $\mathscr{J}$.

2. (10 points) Consider the probability space $([0, 1], \mathscr{B}, P)$, where $\mathscr{B}$ is the Borel σ-algebra on subsets of $[0, 1]$ and P is the Lebesgue measure on $[0, 1]$. Find integrable random variables X and Y defined on the probability space such that $P(X > Y) > 0.5$ and $E[X] < E[Y]$.

3. (10 points) Let X be a standard Gaussian random variable. Show that for any $\alpha > 0$,
$$P(|X| \geq \alpha) \leq \frac{2}{1 + \alpha^2}.$$

4. (10 points) Find two standard Gaussian random variables X and Y such that X and Y are not independent, and $\mathrm{Cov}(X, Y) = 0$.

5. (10 points) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables. Let $S_n = X_1 + \cdots + X_n$. Calculate $E[X_1 | S_n]$.

6. (10 points) Let S_n follow a binomial distribution with parameters n and $\frac{1}{2}$, i.e.,
$$P(S_n = k) = \binom{n}{k} \left(\frac{1}{2}\right)^n, \quad k = 0, 1, 2, \ldots.$$
   Find
$$\lim_{n \to \infty} E\left[\frac{n^2 + n}{(S_n + n)^2}\right].$$

7. (10 points) Let $\{r_n\}_{n \geq 1}$ be an infinite independent fair coin tossing. Let
$$\tau_1 = \inf\{n \geq 2 : r_{n-1} = H, r_n = T\}$$
   and
$$\tau_2 = \inf\{n \geq 4 : r_{n-3} = H, r_{n-2} = T, r_{n-1} = H, r_n = T\},$$
   where H and T denote head and tail outcomes, respectively.
   (a) (5 points points) Compute $E[\tau_1]$.
   (b) (5 points points) Compute $E[\tau_2]$.