Semisimple Lie Algebras

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Why are Lie Algebras Important

1. They are the tangent space of Lie groups
    1. Lie groups are key to studying higher dimensional geometry
    2. Most interesting matrix groups are lie groups
2. You can say a lot about them
    1. Very easy to deal with
    2. Makes proofs much simpler
3. Curcial to modern Physics
    1. Useful in the study of Relativity
    2. Useful in the study of Quantum Mechanics

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Definition

1. *A* **Lie algebra** *is an algebra with a Lie product or bracket*
2. *A* **Lie bracket** *is a function,* $[*, *]$ *such that the following hold*
    1. *It is antisymmetric so* $[a, b] = -[b, a]$
    2. *The Jacobi identity holds so* $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

Remark

We will assume that all elements that we are dealing with are in the general linear group. Therefore, they all have non-zero determinants and hence, inverses. For the Lie algebras that do not staisfy those assumptions there are ways to map them homomorphically into the general linear group, called representations.

Definition

A complex Lie algebra $\mathfrak{g}$ is **reductive** *if there exists a compact matrix Lie gorup K such that*

$$\mathfrak{g} \cong \mathfrak{k}_{\mathbb{C}}$$

A complex LIe algebra $\mathfrak{g}$ is **semisimple** *if it is reductive and the center of $\mathfrak{g}$ is simple Therefore, a semisimple Lie algebra is one that only has 1 as a commutative element and that it is equivalent to the complexification of the Lie algebra of a compact matrix Lie group.*

Remark

These end being the most important ones because any non-semisimple one can be reduced to the direct sum of its center and a simple Lie algebra.

Definition

A **Cartan Subalgebra,**$\mathfrak{h}$**,** *is the center of the group and has the following three properties*

1. *For all* H_1 *and* H_2 *in* $[H_1, H_2] = 0$ *where* [] *is the Lie bracket.*
2. *If for some* $X \in \mathfrak{g}$ *we have* $[H, X] = 0$ *for all* $H \in \mathfrak{h}$ *and* $X \in \mathfrak{h}$
3. *For all* $H \in \mathfrak{h}$, ad_X *is diagonalizable*

Definition

1. *A nonzero element* α *of* $\mathfrak{h}$ *is a* **root** *if there exists a nonzero* $X \in \mathfrak{g}$ *such that*
$$[H, X] = \langle \alpha, H \rangle X$$
for all $H \in \mathfrak{h}$. *The set of all roots is denoted as* R.
2. *If* α *is a root, then the* **root space,** $\mathfrak{g}_\alpha$ *is the space of all* X *in* $\mathfrak{g}$ *for which* $[H, X] = \langle \alpha, H \rangle X$ *for all* H *in* $\mathfrak{h}$. *A nonzero element of* $\mathfrak{g}_\alpha$ *is called a* **root vector** *for* α

Definition

For each $\alpha \in R$, we define a linear map $s_\alpha : \mathfrak{h} \to \mathfrak{h}$ by the formula

$$s_\alpha * H = H - 2\frac{\langle \alpha, H \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

The **Weyl Group** *of R, denoted by W, is the subgroup of $GL(\mathfrak{h})$ generated by the s_α's with $\alpha \in R$*

This map is actually just the reflections of the Cartan subalgebra about the hyperplane orthogonal to α.

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Relationship Amongst Roots

Theorem

For a semisimple Lie algebra, the set R of roots is a finite set of a nonzero elements of a real inner product space E, and R has the following

1. *The roots span E*
2. *If $\alpha \in R$, then $-\alpha \in R$ and the only multiples of α in R are α and $-\alpha$*
3. *If α and β are in R so is $s_\alpha \dot{\beta}$ where*
$$s_\alpha * \beta = \beta - 2\frac{\langle \alpha, H \rangle}{\langle \alpha, \alpha \rangle}\alpha$$
4. *For all α and β in R, the quantity*
$$2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$$
is an integer

Proof.

Proof of point 2.
We can rewrite any element of the Cartan subalgebra as $X = X_1 + iX_2$ with $X_1, X_2 \in \mathfrak{t}$ Let $\overline{X} = X_1 - iX_2$ Since $\mathfrak{t}$ is closed under brackets, if $H \in \mathfrak{t} \subset \mathfrak{k}$ and $X \in \mathfrak{g}$ We have

$$\overline{[H, X]} = [H, X_1] - i[H, X] = [H, \overline{X}]$$

. Because X is a root vector with root α and because of some special properties of the inner product on this space we have

$$[H, \overline{X}] = \overline{[H, X]} = \overline{\langle \alpha, H \rangle X} = -\langle \alpha, H \rangle \overline{X}$$

. Based on the construction of the inner product, the inner product of α and H must be imaginary. Therefore, $[H, \overline{X}] = \langle -\alpha, H \rangle \overline{H}$, making α a root as desired.

□

Geometry Behind Roots

Proposition

Suppose α *and* β *are roots,* α *is not a multiple of* β*, and* $\langle\alpha,\alpha\rangle \geq \langle\beta,\beta\rangle$*. Then oen of the following holds:*

1. $\langle\alpha,\beta\rangle = 0$
2. $\langle\alpha,\alpha\rangle = \langle\beta,\beta\rangle$ *and the angle of* α *and* β *is* $\frac{\pi}{3}$ *or* $\frac{2\pi}{3}$
3. $\langle\alpha,\alpha\rangle = 2\langle\beta,\beta\rangle$ *and the angle of* α *and* β *is* $\frac{\pi}{4}$ *or* $\frac{3\pi}{4}$
4. $\langle\alpha,\alpha\rangle = 3\langle\beta,\beta\rangle$ *and the angle of* α *and* β *is* $\frac{\pi}{6}$ *or* $\frac{5\pi}{6}$

Proof.

Let us assume that α and β are roots and let $m_1 = 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$ and $m_2 = 2\frac{\langle \beta, \alpha \rangle}{\langle \beta \beta \rangle}$. By previous theorem m_1 and m_2 are integers. By definition of inner product we have

$$m_1 m_2 = 4\frac{\langle \alpha, \beta \rangle^2}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} = 4\cos^2\theta$$

. By our initial assumption we also have that

$$\frac{m_2}{m_1} = \frac{\langle \alpha, \alpha}{\langle \beta, \beta \rangle} \geq 1$$

. This restricts the values of $m_1 m_2$ to being 1 2 or 3. If it was 4 then, they would be multiples of one another, which violates our initial assumptions. The specific values of the angles follows with some manipulation from this fact. □

Complete Characterization

Theorem

Every single irreducible root system is isomorphic to exactly one of the following:

1. A_n, $n \geq 1$
2. B_n, $n \geq 2$
3. C_n, $n \geq 3$
4. D_n, $n \geq 4$
5. *One of the exceptional root systems* G_2, F_4, E_6, E_7, *and* E_8.

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$sl(2:\mathbb{C})$

Definition

1. **sl(2:**$\mathbb{C}$ *is the matrix group of 2 by 2 matrices with trace 0 and with the Lie bracket* $[X, Y] = XY - YX$
2. *The* **trace** *of a square matrix is the sum of its eigenvalues.*

The basis vectors to choose are the following:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These matrices give us the commutation relations of

1. $[H, X] = 2X$
2. $[H, Y] = -2Y$
3. $[X, Y] = H$