# Metric Space Topology

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# Outline

- Metric spaces
- Sequences
- Open and closed sets
- Completeness, Compactness, Connectedness
- Theorems from calculus

# Metric Spaces

- A metric space is a pair $(M, d)$ where $M$ is a set of points and $d$ is a metric that satisfies the following
  - Positive definiteness: $d(x, y) \geq 0$. Additionally $d(x, y) = 0$ if and only if $x = y$
  - Symmetry: $d(x, y) = d(y, x)$
  - Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$

# Examples of Metrics

- Consider $\mathbb{R}$ and the metric $d(x, y)$ for $x, y \in \mathbb{R}$,
$$d(x, y) = |x - y|$$
- The Discrete Metric
$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

# Cauchy Sequences

- Definition: Consider a metric space $M$ with the metric $d$. A sequence of points, $a_1, a_2, a_3, ... \in M$ denoted $(a_n)$ is a Cauchy sequence if for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n, k \in \mathbb{N}$ where $n, k \geq N$,

$$d(a_n, a_k) < \epsilon$$

# Convergent Sequences

- Definition: Again consider a metric space $M$ with a metric $d$. A sequence $(p_n)$ converges to the limit $p \in M$ if for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and $n \geq N$,

$$d(p_n, p) < \epsilon$$

- Every convergent sequence is a Cauchy sequence, because as the elements of a sequence converge to some point $b$, they must become closer and closer to one another

# Continuity - Sequences

Definition: A function $f : M \longrightarrow N$ is continuous if it sends convergent sequences in $M$ to convergent sequences in $N$. That is, if $(p_n)$ converges to a limit $p \in M$, then the sequence $(f(p_n))$ converges to the limit $f(p) \in N$

# Closed Sets and Open Sets

Consider a metric space $M$ with metric $d$. Now consider $S$, a subset of $M$. A point $p \in M$ is a limit of $S$ if there is a sequence of points $(p_n)$ in $S$ such that $(p_n)$ converges to $p$.

- A set is closed if it contains all of its limits
- A set is open if for each $x \in S$ there exists an $r > 0$ such that for $y \in M$, if $d(x, y) < r$, then $y \in S$
- The complement of a closed set is an open set, and the complement of an open set is a closed set
- The topology of a metric space $M$ is the collection $\mathcal{T}$ of all open subsets of $M$

# Clopen Sets

Definition: A set is clopen if it is both closed and open

- Consider a metric space $M$, $\emptyset \subset M$. $\emptyset$ is closed since there are no sequences in $\emptyset$, and therefore no limits of sequences in $\emptyset$ that fall outside of it. As well, $\emptyset$ is open since there are no elements in the set, and thus no elements that contradict the condition for the set to be open.
- The complement of $\emptyset$, which is $M$, must then be both open and closed as well.
- Therefore, $M$ and the empty set are clopen sets

# Continuity - Sets

Definition: The following are equivalent conditions for continuity of a function $f : M \longrightarrow N$,

- The closed set condition: The preimage of each closed set in $N$ is a closed set in $M$
- The open set condition: The preimage of each open set in $N$ is an open set in $M$

# Important Notions of a Metric Space

- Completeness
- Compactness
- Connectedness

# Completeness

Definition: A metric space $M$ is complete if each Cauchy sequence in $M$ converges to a limit in $M$

# Compactness

Definition: A subset $A$ of a metric space $M$ is compact if every sequence $(a_n) \in A$ has a subsequence $(a_{n_k})$ that converges to a limit in $A$

# Compactness

Theorem: Every compact set is bounded

Proof: Consider a compact subset $A$ of a metric space $M$. Suppose $A$ is not bounded, then for a sequence $(a_n) \subset A$, for any $p \in M$, we have that $d(a_n, p) \to \infty$ as $n$ goes toward infinity. Since $A$ is compact we know that there is some $(a_{n_k}) \subset (a_n)$ such that $a_{n_k} \to p_0 \in M$. This contradicts the fact that $d(a_n, p) \to \infty$, meaning the set must be bounded.

# Compactness

Theorem: The closed interval $[a, b] \in \mathbb{R}$ is compact

Proof: Consider a sequence $(x_n)$ in $[a, b]$. We can define the set $C$ as,

$$C = \{x \in [a, b] : x_n < x \text{ finitely many times}\}$$

We can say that $a \in C$ since there can be no value in the sequence $(x_n)$ that is less than $a$. $C$ is not empty. As well, $b$ is an upper bound for $C$ since there are no $x \in [a, b]$ that are greater than $b$. There must exist some least upper bound of $C$, $c \in [a, b]$. Suppose there is no subsequence of $(x_n)$ that converges to $c$. Then for some $r > 0$, $x_n < c + r$ finitely many times, since by our assumption the sequence does not converge to $c$. Thus, $c + r \in C$, a contradiction to the fact that $c$ is the least upper bound of $C$. Therefore, there must be a subsequence of $(x_n)$ that converges to $c$ and it follows that $[a, b]$ is compact.

# Compactness

Theorem: The Cartesian product of two compact sets is compact.

Proof: Consider metric spaces $M$ and $N$, where $A \subset M$, $B \subset N$. Suppose $A$ and $B$ are compact. We can define a sequence $(a_n, b_n)$ in $A \times B$. Since $A$ is compact, $(a_n)$ has a subsequence $(a_{n_k})$ that converges to a point $a \in A$. As well, since $B$ is compact, the sequence $(b_{n_k})$ has a subsequence $(b_{n_{k_l}})$ that converges to a point $b \in B$. It follows that, $(a_{n_{k_l}}, b_{n_{k_l}})$ converges to $(a, b) \in A \times B$. Thus, the Cartesian product is compact.

Suppose the Cartesian product of $n$ compact sets is compact. Then the Cartesian product of $n+1$ compact sets,

$$[A_1 \times A_2 \times ... \times A_n] \times A_{n+1}$$

This is the Cartesian product of two compact sets which we know to be compact. By induction the Cartesian product of $m \in \mathbb{N}$ compact sets is compact.

# Compactness

Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}^m$ has a convergent subsequence.

Proof: Every bounded sequence in $\mathbb{R}^m$ can be contained in a box, this box being the Cartesian product of intervals, with $a_i, b_i \in \mathbb{R}$,

$$[a_1, b_1] \times [a_2, b_2] \times ... \times [a_m, b_m]$$

We have shown that each $[a, b]$ is compact, and that the Cartesian product of compact sets is compact, and it follows that a box in $\mathbb{R}^m$ is compact. Thus any sequence in this box must have a convergent subsequence.

# Compactness

Theorem: If $f : M \longrightarrow N$ is continuous and $A$ is a compact subset of $M$, then $f(A)$ is a compact subset of $N$.

Proof: Suppose $(b_n)$ is a sequence in $f(A) = \{f(x) : x \in A\}$. We can assign each $b_n \in f(A)$ with an $a_n \in A$ such that $f(a_n) = b_n$. Since $A$ is compact, there is a subsequence $(a_{n_k})$ that converges to a point $p \in A$. It follows that $f(a_{n_k}) = b_{n_k}$ which converges to $f(p) \in f(A)$. Therefore, for each sequence $(b_n)$ in $f(A)$, there is a subsequence $(b_{n_k})$ that converges to a point $f(p) \in f(A)$. Thus $f(A)$ is compact.

# Connectedness

Definition: Given a metric space $M$, if $M$ has a proper clopen subset $A$, that is $A$ is neither $M$ nor $\emptyset$, then $M$ is disconnected. $M$ is connected if it is not disconnected, that is, there are no proper clopen subsets of $M$

If there are proper clopen subsets of a metric space, $A$ and $A^c$, then we can separate $M$ into nonempty disjoint sets,

$$M = A \dot{\cup} A^c$$

# Connectedness

Theorem: If $M$ is connected and the function $f : M \longrightarrow N$ is continuous and surjective, then $N$ is connected.

Proof: Suppose $A$ is a proper clopen subset $N$. Let $X = \{m \in M : f(m) \in A\}$. $X$ is the preimage of $A$. This preimage $X$ must be clopen since $f$ is continuous. It must be nonempty since $f$ is surjective. It follows that the preimage of $A^c$ must be nonempty as well, implying that $X$ is neither empty nor the set $M$. Thus $X$ is a proper clopen subset of $M$ contradicting the fact that $M$ is connected. It must be that a proper clopen subset of $N$ cannot exist, and thus $N$ is connected.

# Connectedness

Theorem: $\mathbb{R}$ is connected.

Proof: Suppose we have some nonempty clopen subset $U \subset \mathbb{R}$. If we take some $p \in U$, we can make a set,

$$X = \{x \in U : \text{the open interval } (p, x) \subset U\}$$

$X$ is nonempty since $U$ is open. Let $s$ be the supremum of $X$. If $s$ is finite, $s$ must be the least upper bound of $X$ and $s$ is a limit of $U$ since $s$ is the greatest value such that $(p, s) \subset U$. Since $U$ is closed, $s \in U$. Since $U$ is open, for some $r > 0$, we know that the interval $(s - r, s + r) \subset U$. Thus, $s + r \in X$, contradicting the fact that $s$ is the least upper bound of $X$. Therefore it must be that $X$ is unbounded above and the interval $(p, \infty) \subset U$. Repeating this process with the greatest lower bound gives the result that $(-\infty, p) \subset U$. Thus, $U = \mathbb{R}$, there are no proper clopen subsets of $\mathbb{R}$, and $\mathbb{R}$ is connected.

# Theorems from Calculus

- Intermediate Value Theorem
- Extreme Value Theorem (Minimum-Maximum)

# Intermediate Value Theorem

Intermediate Value Theorem: A continuous function defined on an interval $[a, b]$ achieves all intermediate values. If $f(a) = \alpha$, $f(b) = \beta$, and $\gamma$ is given such that $\alpha \leq \gamma \leq \beta$, then there is some $c \in [a, b]$ such that $f(c) = \gamma$.

We can first prove the general intermediate value theorem.

# Intermediate Value Theorem

General Intermediate Value Theorem: Every continuous real valued function defined on a connected domain attains all intermediate values.

Proof: Consider $M$, a connected metric space, and a function $f : M \longrightarrow \mathbb{R}$ which is continuous. As well, $f(a) = \alpha$ and $f(b) = \beta$, where $\alpha < \beta$. Now suppose there exists some $\gamma$, $\alpha < \gamma < \beta$, such that there is no $x \in M$ where $f(x) = \gamma$. Then we can split up $M$ into two open disjoint sets,

$$M = \{x \in M : f(x) < \gamma\} \dot{\cup} \{x \in M : f(x) > \gamma\}$$

This representation contradicts the fact that $M$ is connected, and therefore $f(x)$ must attain $\gamma$. That is, the function attains all intermediate values.

# Intermediate Value Theorem

Theorem: The interval $[a, b]$ is connected.

Proof: Define a function $f : \mathbb{R} \longrightarrow [a, b]$ as follows,

$$f(x) = \begin{cases} a, & x \leq a \\ x, & a < x < b \\ b, & x \geq b \end{cases}$$

This is a surjective continuous function from $\mathbb{R}$ to a closed interval $[a, b] \subset \mathbb{R}$. We have shown that $\mathbb{R}$ is connected and from an earlier theorem, this implies that $[a, b]$ is also connected.

# Intermediate Value Theorem

We can now prove the Intermediate Value Theorem that we initial presented.

Proof: Apply the General Intermediate Value Theorem to the connected domain $[a, b]$.

# Extreme Value Theorem

Extreme Value Theorem: A continuous function $f$ defined on an interval $[a, b]$ takes on absolute minimum and absolute maximum values, that is for some $x_0, x_1 \in [a, b]$ and for all $x \in [a, b]$, $f(x_0) \leq f(x) \leq f(x_1)$.

# Extreme Value Theorem

We can use some previous theorems to prove the Extreme Value Theorem.

Proof: We have shown that $[a, b]$ is compact, compact sets are bounded, and that the continuous image of a compact set is compact. Therefore a continuous function $f$ defined on $[a, b]$ has bounds $m, M$ such that for all $x \in [a, b]$, $m \leq f(x) \leq M$. Let $M, m$ be the supremum and infimum of the set $\{f(x) : x \in [a, b]\}$ respectively. Thus, there is a sequence $(x_n) \in [a, b]$ such that $f(x_n) \to M$. By compactness, there exists a subsequence $(x_{n_k}) \to x_1 \in [a, b]$ and $f(x_{n_k}) \to f(x_1)$. We also have that $f(x_{n_k}) \to M$. Therefore, $f(x_1) = M$. By symmetry, $f(x_0) = m$ for some $x_0 \in [a, b]$. Thus, we have for all $x \in [a, b]$, $f(x_0) \leq f(x) \leq f(x_1)$.

# Textbook

The following is the textbook that we used throughout the program, and where I referenced to complete the proofs in this presentation.

Pugh, Charles C. *Real Mathematical Analysis*. 2nd ed., Springer, 2017.

Thank you for listening to my presentation, and thank you to my mentor Geoff Lindsell and the organizers of the Directed Reading Program