Abstract Algebra Prelim Jan. 2021

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. (**10 pts**)
   (a) (**5 pts**) If G is a group and $g, h \in G$ have orders m and n with $\gcd(m, n) = 1$ and $gh = hg$ then prove gh has order mn. (This has counterexamples if $\gcd(m, n) > 1$ or $gh \neq hg$!)
   (b) (**5 pts**) The prime factorization of 2021 is $43 \cdot 47$. Use (a) and the Sylow theorems to prove all groups of order 2021 are cyclic.

2. (**10 pts**) Let $G = \mathrm{GL}_2(\mathbf{F}_3)$ act on the set X of all 1-dimensional subspaces of $\mathbf{F}_3^2$:

$$X = \left\{ \mathbf{F}_3\begin{pmatrix}1\\0\end{pmatrix}, \ \mathbf{F}_3\begin{pmatrix}0\\1\end{pmatrix}, \ \mathbf{F}_3\begin{pmatrix}1\\1\end{pmatrix}, \ \mathbf{F}_3\begin{pmatrix}2\\1\end{pmatrix} \right\}$$

   and for $A \in G$ and a 1-dimensional subspace $L = \mathbf{F}_3\mathbf{v}$ in X, set $A(L) = \mathbf{F}_3 \, A(\mathbf{v})$.
   (a) (**4 pts**) Prove this action of G on X has one orbit.
   (b) (**4 pts**) Compute the stabilizer subgroup of $\mathbf{F}_3\binom{1}{0}$.
   (c) (**2 pts**) Compute $|G|$ (you may use (a) and (b) for this, or another method.)

3. (**10 pts**)
   (a) (**3 pts**) Define what irreducibility means in an integral domain.
   (b) (**7 pts**) Prove Eisenstein's irreducibility criterion for monic polynomials in $\mathbf{Z}[x]$: if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is monic in $\mathbf{Z}[x]$ and there is a prime p such that $p \mid a_j$ for $j < n$ and $p^2 \nmid a_0$, then $f(x)$ is irreducible in $\mathbf{Z}[x]$.

4. (**10 pts**)
   (a) (**5 pts**) In the ring $\mathbf{Z}[x]$, compute the index of the ideal $(2, x^2)$ in $\mathbf{Z}[x]$ and determine a maximal ideal that contains $(2, x^2)$.
   (b) (**5 pts**) In the ring $\mathbf{Z}[\sqrt{5}]$, show the principal ideal (3) is maximal.

5. (**10 pts**) On the ring $V = \mathbf{R}[x]/(x^2 - 4)$, viewed as a 2-dimensional real vector space, there is an inner product defined by $\langle f, g\rangle = f(2)g(2) + f(-2)g(-2)$. (You don't have to check this.)
   (a) (**4 pts**) For each $h \in V$, let $L_h \colon V \to V$ by $L_h(f) = hf$. Prove L_h is self-adjoint for the inner product above.
   (b) (**6 pts**) Find the matrix representation for L_{1+3x} with respect to the basis $\{1, x\}$ of V and use it to find a basis of eigenvectors in V for L_{1+3x}. (The matrix with respect to $\{1, x\}$ is not symmetric, which is not a problem since 1 and x aren't eigenvectors of L_{1+3x}.)

6. (**10 pts**) Give examples as requested, with justification.
   (a) (**2.5 pts**) A group isomorphism $A_4 \rtimes_\varphi \mathbf{Z}/(2) \to S_4$ for some φ.
   (b) (**2.5 pts**) An irreducible factorization of 13 in $\mathbf{Z}[\sqrt{3}]$.
   (c) (**2.5 pts**) A ring isomorphism $\mathbf{R}[x]/(x^2 + 3) \cong \mathbf{C}$.
   (d) (**2.5 pts**) A nonzero element in the dual space $(\mathbf{R}^2)^*$ that is in the kernel of the dual to the linear map $\left(\begin{smallmatrix}1 & 3\\2 & 6\end{smallmatrix}\right) \colon \mathbf{R}^2 \to \mathbf{R}^2$. (Write an answer as a linear combination of the dual basis of the standard basis of $\mathbf{R}^2$.)