Abstract Algebra Prelim Aug. 2021

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. (**10 pts**)

   (a) (**4 pts**) Show there is exactly one nontrivial semidirect product $\mathbf{Z}/(4) \rtimes \mathbf{Z}/(4)$ and write down the resulting group law on $(a,b)(c,d)$ in this semidirect product.

   (b) (**6 pts**) Let $G = \mathbf{Z}/(4) \times \mathbf{Z}/(4)$ and $H = \mathbf{Z}/(4) \rtimes \mathbf{Z}/(4)$, where the nontrivial semidirect product H comes from (a). In both G and H, show $(x,y)^4 = (0,0)$ for all (x,y), so all elements of G and H have order dividing 4. Then determine all elements of order 2 in each group and explain why that implies G and H have the same number of elements of each order. (This is interesting since $G \not\cong H$, as one group is abelian and one is not.)

2. (**10 pts**) Let R be a commutative ring (with identity).

   (a) (**2 pts**) Define the notion of an ideal in R.

   (b) (**5 pts**) For each $a \in R$, show $(a) := \{ra : r \in R\}$ is an ideal in R and each ideal in R containing a contains (a).

   (c) (**3 pts**) Show that if $R \neq 0$ and the only ideals in R are (0) and R then R is a field.

3. (**10 pts**) Let R be a commutative ring (with identity), $R \neq 0$, and N be the set of non-units in R. Assume that for all $u, v \in R$ such that $u + v = 1$, at least one of u or v is in $R^\times$.

   (a) (**6 pts**) Show that N is an ideal in R and $N \neq R$.

   (b) (**4 pts**) Show every proper ideal in R is contained in N and explain why N is the only maximal ideal in R.

4. (**10 pts**) For all $n \times n$ real matrices A and B, where $n \geq 1$, define $\langle A, B\rangle = \mathrm{Tr}(AB^\top)$.

   (a) (**4 pts**) Use properties of matrix transposes and the trace on matrices to show $\langle \cdot, \cdot \rangle$ is an inner product on $\mathrm{M}_n(\mathbf{R})$.

   (b) (**3 pts**) Let $E_{ij} \in \mathrm{M}_n(\mathbf{R})$ have (i,j)-entry 1 and its other entries all equal 0. Their multiplicative relations are $E_{ij}E_{jl} = E_{il}$ and $E_{ij}E_{kl} = O$ if $j \neq k$ (no need to prove this). Show the matrices E_{ij} for all i, j are an orthonormal basis of $\mathrm{M}_n(\mathbf{R})$ with respect to $\langle \cdot, \cdot \rangle$.

   (c) (**3 pts**) Prove that symmetric and skew-symmetric matrices in $\mathrm{M}_n(\mathbf{R})$ are orthogonal for this inner product: if $A^\top = A$ and $B^\top = -B$, then $\langle A, B\rangle = 0$. (This doesn't need (b).)

5. (**10 pts**) Let V be an n-dimensional F-vector space ($n \in \mathbf{Z}^+$) with dual space V^*.

   (a) (**2 pts**) For a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ of V, define the dual basis $\mathcal{B}^* = \{v_1^*, \ldots, v_n^*\}$ in V^*.

   (b) (**5 pts**) Prove that $\mathcal{B}^* = \{v_1^*, \ldots, v_n^*\}$ from (a) is a basis of V^*.

   (c) (**3 pts**) Let $V = \{a + bx + cx^2 : a, b, c \in F\}$ be the F-vector space of polynomials of degree at most 2 (and the zero polynomial), and let $\mathcal{B} = \{v_1, v_2, v_3\}$ where $v_1 = 1$, $v_2 = x - 1$, and $v_3 = (x-1)^2$. Write $x^2 + 1$ as a linear combination of $\mathcal{B}$ and compute $v_2^*(x^2 + 1)$.

6. (**10 pts**) Give examples as requested, with justification.

   (a) (**2.5 pts**) For prime p, give a list of all *abelian* groups of order p^4 up to isomorphism.

   (b) (**2.5 pts**) Two permutations in some S_n that have the same order and are not conjugate.

   (c) (**2.5 pts**) An irreducible cubic polynomial in $\mathbf{Z}[x]$ that is not an Eisenstein polynomial.

   (d) (**2.5 pts**) A nonprincipal ideal in $\mathbf{Z}[x]$.