COMPLEX ANALYSIS PRELIM

AUGUST 2021

Notation and conventions:

- Denote by $\mathbb{C}$ the complex plane and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk.
- A *region* means a nonempty connected open set.
- The terminology *analytic* function and *holomorphic* function may be used interchangeably.

Problem 1. How many distinct $z \in \mathbb{D}$ satisfy $\dfrac{z^3 + 5z^2 + 1}{z + 6} = 0$?

Problem 2. Let $f(z) = \dfrac{1}{z^2 + 1}$, $\gamma_r = \{z \in \mathbb{C} : |z + i + 1| = r\}$. Consider $g(r) = \displaystyle\int_{\gamma_r} f(z)dz$, where $r > 0$, and each γ_r is positively oriented. Find the domain of definition and the values of $g(r)$.

Problem 3. Describe a holomorphic isomorphism between regions $U = \{z : |z| < 1, Im(z) > 0\}$ and $V = \{z : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$.

Problem 4. Let f be a rational function such that all its poles are contained in $\mathbb{D}$. Prove that f has an antiderivative outside of $\mathbb{D}$ if and only if the rational function $g(z) = \dfrac{1}{z^2} f\left(\dfrac{1}{z}\right)$ has residue 0 at $z = 0$.

Problem 5. Let f be a holomorphic function in a region. We say z_0 in the region is a local maximum for $|f|$ if $|f(z_0)| \geq |f(z)|$ for all z near z_0, and z_0 is a local minimum for $|f|$ when that inequality is reversed.

(a) Prove the strong maximum principle: f is constant if and only if $|f|$ has a local maximum.
(b) Is it true that f is constant if and only if $|f|$ has a local minimum? Give a proof or a counterexample.

Problem 6.

(a) Give definitions of a harmonic conjugate and of a normal family.
(b) Assume that u is a harmonic function in a simply connected region and v is its harmonic conjugate. Is it true that u is bounded if and only if v is bounded? Give a proof or a counterexample.
(c) Assume that $\mathcal{U} = \{u_n\}$ is a family of harmonic functions in a simply connected region and $\mathcal{V} = \{v_n\}$ is the family of their harmonic conjugates. Is it true that $\mathcal{U}$ is normal if and only if $\mathcal{V}$ is normal? Give a proof or a counterexample.
(d) Answer the same question as in (c) with the extra condition: there is a point z such that $u_n(z) = v_n(z) = 0$ for all n.