Topology Prelim, August 2021

1. A topological space X is said to be metrizable if its topology is generated by some metric on X. Does every compact metrizable space X have a countable basis? Prove your assertion.

2. Suppose $q : X \to Y$ is an open quotient map. Prove that Y is Hausdorff if and only if the set $\mathfrak{R} = \{(x_1, x_2) : q(x_1) = q(x_2)\}$ is closed in $X \times X$.

3. Prove that a space X is contractible if and only if every map $f : X \to Y$, for an arbitrary Y, is null-homotopic.

4. Let X be the space obtained from $\mathbb{R}^3$ by removing the circle $C = \{(0, y, z) \in \mathbb{R}^3 : y^2 + z^2 = 4\}$. Compute $\pi_1(X)$.

5. Let $p : X \to Y$ be a covering map and Y be path-connected and locally path-connected. If $A \subseteq X$ is a path component of X, is $p_A : A \to Y$ a covering map? Prove your assertion. Here p_A is obtained by restricting p on A.

6. A topological space X is said to be **normal** if it is Hausdorff and for every pair of disjoint closed subsets $A, B \subseteq X$, there exist disjoint open subsets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$. Prove that if X is compact and Hausdorff then X is normal.