Some Topics in Algebraic Combinatorics

Hanzhang Yin

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Outline

1. Introduction to the combinatoric with "coloring" $1 \times n$ rectangles
2. Definition of graphs and walks
3. Counting walks on graphs
4. Probability matrix of a graph

Example Combinatorial Problems

- n-colorings of 1×5 boards.

- Squares are colored with letters.

A	B	C	D	E

- Rotating the board 180°, gives a new coloring.

A	B	C	D	E

$\xrightarrow{180^\circ}$

E	D	C	B	A

- We define two colorings are "the same" if rotating one results in the other.
- Special Case

A	B	C	B	A

n-Colorings of 1×5 Board

- Goal: Count the number of unique colorings with 180° flips.

n	n	n	n	n

n^5

A	B	C	B	A

n^3

- $n^5 - n^3$ The number of colorings which don't equal their 180° rotation.

A	B	C	D	E

$\cong$

E	D	C	B	A

(One equivalence class)

- $\frac{1}{2}(n^5 - n^3)$ (The number of equivalence class of colorings which don't equal their 180° rotation.)

n-Colorings of 1×5 Board (Continued)

- $\frac{1}{2}(n^5 - n^3)$ (The number of different equivalence classes of n-colorings which don't equal their 180° rotation.)
- n^3 (The number of different equivalence classes of n-colorings which equal their 180° rotation.)
- $\frac{1}{2}(n^5 - n^3) + n^3$ (The total number of different equivalence classes of n-colorings.)
- Note: This argument can be generalized to n-colorings of $1 \times k$ board.

Multiset

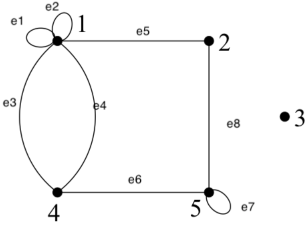
- Given a finite set S and integer $k \geq 0$.
- $\binom{S}{k}$ denotes the set of k-element subsets of S.
- e.g. $S = \{1, 2, 3\}$ and $k = 2$
- $\binom{S}{2} = \{12, 13, 23\}$

Multiset

- A multiset is a set with repeated elements
- e.g. $\{1, 1, 2, 2, 3, 3\}$
- $\{1, 2, 1, 3, 2, 3\} = \{1, 1, 2, 2, 3, 3\}$
- $\left(\binom{S}{k}\right)$ denotes the set of k-elements multisets on S.
- $S = \{1, 2, 3\}$ and $k = 2$
- $\binom{S}{k} = \{12, 13, 23\}$, $\left(\binom{S}{k}\right) = \{11, 22, 33, 12, 13, 23\}$

Graphs

- A (finite) graph G consists of a vertex set $V = \{v_1, v_2, v_3, \cdots, v_p\}$ and edges set $E = \{e_1, \cdots, e_q\}$ with a function $\psi : E \to (\binom{V}{2})$



Graphs

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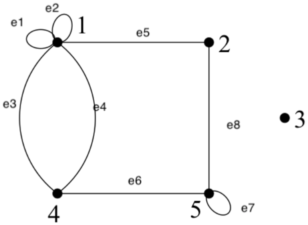
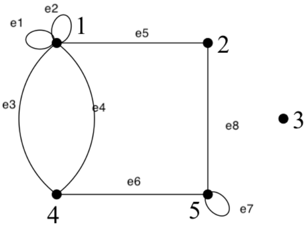


- $V(vertex) = \{1, 2, 3, 4, 5\}$ and $E(edge) = \{e1, e2, e3, e4, e5, e6, e7, e8\}$
- $(\binom{V}{2}) = \{11, 22, 33, 44, 55, 12, 13, 14, 15, 23, 24, 25, 34, 35, 45\}$

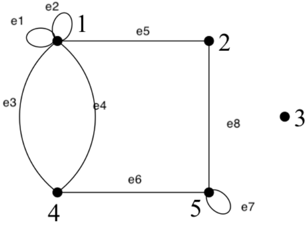
Graphs



- $E(edge) = \{e1, e2, e3, e4, e5, e6, e7, e8\}$
- $(\binom{V}{2}) = \{11, 22, 33, 44, 55, 12, 13, 14, 15, 23, 24, 25, 34, 35, 45\}$
- e.g. $\psi(e1) = \psi(e2) = 11$ (e1, e2 are called loops)
- $\psi(e3) = \psi(e4) = 14$ (there is a multiple edge between 1 and 4)

Adjacency Matrix of the graph G

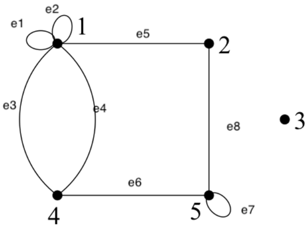
- p is the number of vertices in the graph.
- The adjacency matrix of the graph G is the $p \times p$ matrix $A = A(G)$, whose (i, j)-entry a_{ij} is equal to the number of edges incident to v_i and v_j.



- $A(G) = \begin{bmatrix} 2 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$

Walks

- A walk in G of *length* ℓ from vertex u to vertex v is a sequence $v_{a_1} = u, e_{a_1}, v_{a_2}, e_{a_2}, \cdots, v_{a_\ell}, e_{a_\ell}, v_{a_{\ell+1}} = v$



- A walk in G of *length* 1 from vertex 1 to vertex 2 is a sequence $1, e5, 2$
- A walk in G of *length* 2 from vertex 1 to vertex 2 could be the sequence $1, e2, 1, e5, 2$ and sequence $1, e1, 1, e5, 2$.

Counting Walks

Goal:Count the number of walks from vertex u to vertex v.

Theorem

For any integer $\ell \geq 1$, the (i,j)-entry of the matrix $A(G)^\ell$ is eq al to the number of walks from v_i to v_j in G of length ℓ.

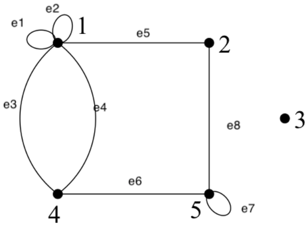
Sketch of proof

Let $A = (a_{ij})$. The (i,j)-entry of $A(G)^\ell$ is given by

$$(A(G)^\ell)_{ij} = \sum a_{ii_1} a_{i_1 i_2} \cdots a_{i_{\ell-1} j}$$

where the sum ranges over all sequences $(i_1, \cdots, i_{\ell-1})$

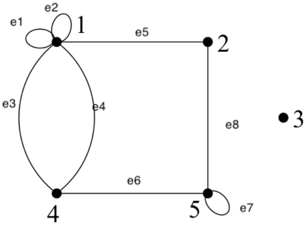
Example



- $\ell = 2$
- For each sequence of ℓ, vertices starting at i and ending at j, there are a_{ii} walks of length one from vertex i to i and then a_{ii_1} walks of length one from a_i to a_{i_1}, and so on, after ℓ steps we arrive at j, then sum over all such sequences
- $$(A(G)^2)_{21} = a_{21}a_{11} + a_{22}a_{21} + a_{23}a_{31} + a_{24}a_{41} + a_{25}a_{51}$$

$$(A(G)^2)_{21} = 2 \cdot 1 + 0 \cdot 1 + 0 \cdot 0 + 0 \cdot 2 + 1 \cdot 0 = 2$$

Example (Continued)

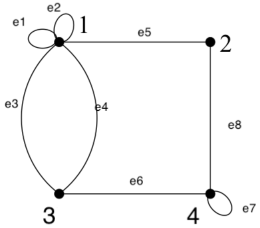


$\ell = 2$

$$A(G)^2 = A(G) = \begin{bmatrix} 2 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 9 & 2 & 0 & 4 & 3 \\ 2 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 0 & 5 & 1 \\ 3 & 1 & 0 & 1 & 3 \end{bmatrix}$$

$$(A(G)^2)_{21} = 2$$

Example (Continued)



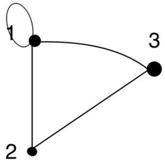
- $\ell = 2$

$$A(G)^2 = A(G) = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 9 & 2 & 4 & 3 \\ 2 & 2 & 3 & 1 \\ 4 & 3 & 5 & 1 \\ 3 & 1 & 1 & 3 \end{bmatrix}$$

$$(A(G)^2)_{21} = 2$$

$A^\ell = U.diag(\lambda_1^\ell, ..., \lambda_p^\ell)U^{-1}$

- An easier way to count the number of walks
- A real symmetric $p \times p$ matrix M has p linearly independent real eigenvectors.



$$A(G) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\lambda_1 = 1 + \sqrt{2}$, $\lambda_2 = -1$, $\lambda_3 = 1 - \sqrt{2}$

$A^\ell = U.diag(\lambda_1^\ell, ..., \lambda_p^\ell)U^{-1}$

$$A(G) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\lambda_1 = 1 + \sqrt{2}$$

$$\lambda_2 = -1$$

$$\lambda_3 = 1 - \sqrt{2}$$

$$v_1 = (\sqrt{2}, 1, 1)$$

$$v_2 = (0, -1, 1)$$

$$v_3 = (-\sqrt{2}, 1, 1)$$

$A^\ell = U.diag(\lambda_1^\ell, ..., \lambda_p^\ell)U^{-1}$

- **Goal**: Diagonalize A

$$A(G) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$diag(\lambda_1^\ell, ..., \lambda_p^\ell) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1-\sqrt{2} & 0 \\ 0 & 0 & 1+\sqrt{2} \end{bmatrix}^\ell$$

$$U^{-1} = U^T = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$A^\ell = U.diag(\lambda_1^\ell, ..., \lambda_p^\ell)U^{-1}$

$$A(G)^\ell = (U \cdot diag(\lambda_1, ..., \lambda_p) \cdot U^{-1})^\ell$$

$$A(G)^\ell = (U \cdot diag(\lambda_1, ..., \lambda_p) \cdot U^{-1}) \dots (U \cdot diag(\lambda_1, ..., \lambda_p) \cdot U^{-1})$$

$$A(G)^\ell = U \cdot diag(\lambda_1, ..., \lambda_p)^\ell \cdot U^{-1}$$

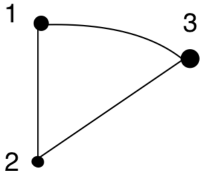
$$A(G)^\ell = U \cdot diag(\lambda_1^\ell, ..., \lambda_p^\ell) \cdot U^{-1}$$

$$= \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1-\sqrt{2} & 0 \\ 0 & 0 & 1+\sqrt{2} \end{bmatrix}^\ell \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

The complete graph K_p

K_p is a graph with vertex set $V = \{v_1, ..., v_p\}$, and one edge between any two distinct vertices.

- K_p has p vertices and $\binom{p}{2} = \frac{1}{2}p(p-1)$ edges.
- Example



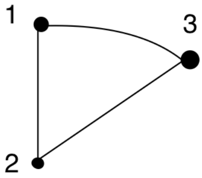
- $V = \{1, 2, 3\}$. K_3 has 3 vertices and $\binom{3}{2} = \frac{1}{2}3(3-1) = 3$ edges.

Closed walks

- A *closed walk* of G is a walk that ends where it begins.
- The number of closed walks of length ℓ in K_p from some vertex v_i to itself is given by

$$(A(K_p)^{\ell})_{ii} = \frac{1}{p}((p-1)^{\ell} + (p-1)(-1)^{\ell})$$

- Example: Complete Graph K_3



$$(A(K_3)^1)_{ii} = \frac{1}{3}((3-1)^1 + (3-1)(-1)^1) = 0$$

$$(A(K_3)^2)_{ii} = \frac{1}{3}((3-1)^2 + (3-1)(-1)^2) = 2$$

Algebraic Proof

- J denotes the $p \times p$ matrix of all $1's$ and I is the identity matrix.
- $J_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \ I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $A(K_p) = J - I$
- $A(K_3) = J_3 - I_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
- Binomial theorem

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Algebraic Proof

$$A(K_p)^\ell = (J - I)^\ell = \sum_{k=0}^{\ell} \binom{\ell}{k} J^k (-I)^{\ell-k}$$

$$J^k = p^{k-1} J (mathematical\ induction)$$

$$A(K_p)^\ell = (J - I)^\ell = \sum_{k=1}^{\ell} (-1)^{\ell-k} \binom{\ell}{k} p^{k-1} J + (-1)^\ell I$$

Binomial theorem again,

$$(J - I)^\ell = \frac{1}{p}((p-1)^\ell - (-1)^\ell) J + (-1)^\ell I$$

Algebraic Proof

$$A(K_p)^{\ell}_{ij} = \frac{1}{p}((p-1)^{\ell} - (-1)^{\ell})$$

$$A(K_p)^{\ell}_{ii} = \frac{1}{p}((p-1)^{\ell} + (p-1)(-1)^{\ell})$$

The total number of walks of length ℓ in K_p.

$$\sum_{i=1}^{p}\sum_{j=1}^{p}(A(K_p)^{\ell})_{ij} = p(p-1)^{\ell}$$

- Note: We will sum over all the walks instead of just closed walks.

  Proof:

$$(p^2-p)A(K_p)^{\ell}_{ij} + pA(K_p)^{\ell}_{ii} =$$

$$(p^2-p)\frac{1}{p}((p-1)^{\ell}-(-1)^{\ell}) + p\cdot\frac{1}{p}((p-1)^{\ell}+(p-1)(-1)^{\ell}) =$$

$$p(p-1)^{\ell}$$

Probability matrix

Let $M = M(G)$ be the matrix whose rows and columns are indexed by the vertex $\{v_1, \cdots, v_p\}$ of G, and whose (u,v)-entry is given by

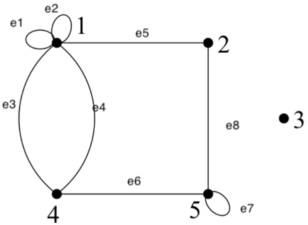
$$M_{uv} = \frac{\mu_{uv}}{d_u}$$

μ_{uv} - the number of edges between u and v.

d_u - the number edges incident to u

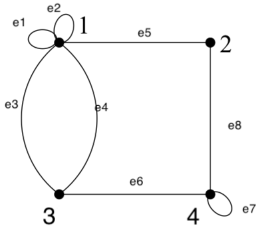
M_{uv} - the probablity that is one starts at u, then the next step will be to v.

Example



- When $u = 2$, $d_u = 2$
- When $u = 2$ and $v = 1$, $\mu_{uv} = 1$
- (2,1)-entry is $M_{21} = \frac{\mu_{21}}{d_2} = \frac{1}{2}$
- note: $d_3 = 0$ so we can get rid of isolated point 3.

Example



- $$M(G) = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

  $$A(G) = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Theorem: Let G be a finite graph. Then the probablity matrix $M = M(G)$ is diagonalizable and has only real eigenvalues.

- Let D be the diagonal matrix whose rows and columns are indexed by the vertices of G, with $D_{vv} = \sqrt{d_v}$
- Then

$$(DMD^{-1})_{uv} = \sqrt{d_u} \cdot \frac{\mu_{uv}}{d_u} \cdot \frac{1}{\sqrt{d_v}}$$

$$(DMD^{-1})_{uv} = \frac{\mu_{uv}}{\sqrt{d_u d_v}}$$

- DMD^{-1} is a symmetric matrix.
- M is diagonalizable and has only real eigenvalues.

Thank you

Citation:
Stanley, Richard, "Topics in Algebraic Combinatorics"