Measure Theory and Applications to Probability and Signal Filtering

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The downsides of the Riemann Integral

- The Riemann Integral only works with bounded functions
- Define $f : [0,1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

- When one uses the definition for Riemann Integrability we find $L(f, P, [0,1]) = 0$ and $U(f, P, [0,1]) = 1$ which implies $f(x)$ is not Riemann Integrable.

Length of Intervals

- How can we get the length of an interval?
- Definition: The length L(I) of an open interval I is defined by

$$L(I) = \begin{cases} \text{b-a} & \text{if } I = (a, b) \text{ for some } a, b \in \mathbb{R} \text{ with a} \leq \text{b} \\ 0 & \text{if } I = \emptyset \\ \infty & \text{if } I = (-\infty, a) \text{ or } I = (a, \infty) \text{ for some } a \in \mathbb{R} \\ \infty & \text{if } I = (-\infty, \infty) \end{cases}$$

Using length to define Outer Measure

Definition:

The outer measure of $|A|$ of a set $A \subset \mathbb{R}$ is defined by

$$|A| = inf \left\{ \sum_{k=1}^{\infty} L(I_k) : I_1, I_2, \ldots \text{ are open intervals such that } A \subset \cup_{k=1}^{\infty} I_k \right\}$$

Heine-Borel Theorem

Definition:

Suppose $A \subset \mathbb{R}$ then call the collection $\mathcal{C}$ of open subsets of $\mathbb{R}$ an open cover of A if A is contained in the union of all the sets in $\mathcal{C}$. $\mathcal{C}$ has a finite subcover of A if A is contained in a finite list of sets in $\mathcal{C}$.

Heine-Borel Theorem:

Every open cover of a bounded closed subset of $\mathbb{R}$ has a finite subcover.

Important Properties of Outer Measure

- Countable subsets of R have outer measure 0!
- Outer Measure preserves order, i.e let
  A and B be subsets of $\mathbb{R}$ such that $A \subset B$ then $|A| \leq |B|$
- Outer Measure is translation invariant! That is if $t \in \mathbb{R}$ and $A \subset \mathbb{R}$ then
  $t + A = \{ t + a : a \in A \}$ then $|t + A| = |A|$
- There always exists disjoint subsets of A and B of $\mathbb{R}$ such that $|A \cup B| \neq |A| + |B|$

Applications of Outer Measure

- On a closed interval, $a, b \in \mathbb{R}$, $a \leq b$, then $|[a, b]| = b - a$. Done with Heine-Borel Theorem.
- Every interval in $\mathbb{R}$ containing at least two distinct elements is uncountable.
- Originally shown by Georg Cantor, but with a very lofty proof, the proof for this is much simpler.

σ-Algebras

Definition: σ-Algebras

Let X be a set and $\mathcal{S}$ be a set of subsets of X. Then $\mathcal{S}$ us considered a σ-Algebra if:

- $\emptyset \in \mathcal{S}$
- if $E \in \mathcal{S}$ then $X \setminus E \in \mathcal{S}$
- if $E_1, E_2 ...$ is a sequence of events in $\mathcal{S}$ then $\cup_{k=1}^{\infty} E_k \in \mathcal{S}$

σ-Algebras closed under countable intersection

Suppose $\mathcal{S}$ is a σ-Algebra on X. Then,

- $X \in \mathcal{S}$
- if $D, E \in \mathcal{S}$, then $D \cup E \in \mathcal{S}$ and $D \cap E \in \mathcal{S}$ and $D \setminus E \in \mathcal{S}$
- if $E_1, E_2, ...$ is a sequence of events in $\mathcal{S}$ then $\cap_{k=1}^{\infty} E_k \in \mathcal{S}$

Measurable Spaces

Definition: Measurable Spaces and Sets

- A measurable space is simply an ordered pair $(X, \mathcal{S})$ where X is a set and $\mathcal{S}$ is a σ-Algebra on X.
- An event in $\mathcal{S}$ is called an $\mathcal{S}$-measurable set

Borel Sets

Smallest σ-algebra

If X is a set and $\mathcal{A}$ is a set of subsets of X, then the intersection of σ-algebras on X that contain $\mathcal{A}$ is also a σ-algebra on X.

Definition: Borel Set

The smallest σ-algebra on $\mathbb{R}$ containing all open subsets of $\mathbb{R}$ is called the collection of Borel subsets of $\mathbb{R}$. Any element of this σ-algebra is called a Borel Set.

- A subset $[-\infty, \infty]$ is called a Borel set

- Every closed subset of $\mathbb{R}$ is a Borel set since every closed subset of $\mathbb{R}$ is the complement of an open subset of $\mathbb{R}$
- Every countable subset of $\mathbb{R}$ is a Borel set, to see this let a set $A = x_1, x_2, \ldots$ then $B = \cup_{k=1}^{\infty} \{x_k\}$ is a Borel set since each $\{x_k\}$ is closed
- If $f : \mathbb{R} \longrightarrow \mathbb{R}$ any set of points where f is continuous is the intersection of open sets and thus a Borel set

Inverses and their Properties

Definition: Inverse Images

If $f : X \longrightarrow Y$ is a function and $A \subset Y$ then the set $f^{-1}(A)$ is defined by,

$$f^{-1}(A) = \{x \in X : f(x) \in A\}$$

Properties of Inverses

- Suppose $f : X \longrightarrow Y$ and $g : Y \longrightarrow W$ are functions, then

$$(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)) \text{ for every } A \subset W$$

- Suppose $f : X \longrightarrow Y$, then
    - $f^{-1}(Y \setminus A) = X^{-1}(A)$ for every $A \subset Y$
    - $f^{-1}(\cup_{A \in \mathcal{A}}(A)) = \cup_{A \in \mathcal{A}} f^{-1}(A)$ for every set $\mathcal{A}$ of subsets of $\mathcal{Y}$)
    - $f^{-1}(\cap_{A \in \mathcal{A}}(A)) = \cap_{A \in \mathcal{A}} f^{-1}(A)$ for every set $\mathcal{A}$ of subsets of $\mathcal{Y}$)

Measurable Functions and their Conditions

Definition: Measurable Functions

Suppose $(X, \mathcal{S})$ is a measurable space then the function $f : X \longrightarrow \mathbb{R}$ is called $\mathcal{S}$-measurable if $f^{-1}(B) \in \mathcal{S}$ for every Borel set $B \in \mathbb{R}$

Definition: Borel Measurable Function

Suppose $X \subset \mathbb{R}$, a function $f : X \longrightarrow \mathbb{R}$ is called Borel Measurable if $f^{-1}(B)$ is a Borel set for every Borel set $B \subset \mathbb{R}$

- Every continuous function is Borel measurable!

Measures

Definition: Measure

Suppose X is a set and $\mathcal{S}$ is a σ-algebras on X. A measure on $(X, \mathcal{S})$ is a function $\mu : \mathcal{S} \longrightarrow [0, \infty]$ such that $\mu(\emptyset) = 0$ and

$$\mu(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k)$$

for every disjoint sequence $E_1, E_2, ...$ of sets in $\mathcal{S}$

Properties of Measure

Definition: Measure Space

A measure space is the ordered triple $(X, \mathcal{S}, \mu)$ where X is a set $\mathcal{S}$ is a σ-algebras on X, and μ is a measure on $(X, \mathcal{S})$

Properties of Measure

- Measure preserves order, $(D \subset E)$ then $\mu(D) \leq \mu(E)$
- $(X, \mathcal{S}, \mu)$ is a measure space and $E_1, E_2, ... \in \mathcal{S}$. Then, $\mu(\cup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} \mu(E_k)$
- $\mu(D \cup E) = \mu(D) + \mu(E) - \mu(D \cap E)$, $\mu(D \cap E) \leq \infty$

Connection Between Outer Measure and Lebesgue Measure

Outer Measure and Lebesgue Measure

Outer measure is a measure on $(\mathbb{R}, \mathcal{B})$ where $\mathcal{B}$ is a σ-algebra of Borel subsets of $\mathbb{R}$

Definition: Lebesgue Measure

Lebesgue measure is the measure on $(\mathbb{R}, \mathcal{B})$ where $\mathcal{B}$ is the σ-algebra that assigns each Borel set to its outer measure.

Definition: Lebesgue Measurable Set

A set $A \subset \mathbb{R}$ is called Lebesgue measurable if there exists a Borel set $B \subset A$ such that $|A \setminus B| = 0$ is the measure on $(\mathbb{R}, \mathcal{B})$ where $\mathcal{B}$ is the σ-algebra that assigns each Borel set to its outer measure.

Convergence of Measurable Functions

Definition: Pointwise and Uniform Convergence

Let X be a set with $f_1, f_2, \ldots$ being a sequence of functions from X to $\mathbb{R}$ and f is a function from X to $\mathbb{R}$.

- The sequence $f_1, f_2, \ldots$ converges pointwise on X to f if $\lim_{k \longrightarrow \infty} f_k(x) = f(x)$
- The sequence $f_1, f_2, \ldots$ converges uniformly on X to f if for every $\epsilon > 0$ there exists a $n \in \mathbb{Z}^+$ such that $|f_k(x) - f(x)| < \epsilon$ for all integers $k \geq n$ and all $x \in$ X

Simple Functions and Approximations with them

- A function is simple if it takes only finitely many values
- We can approximate functions by simple functions!
  - Let each f_k be a simple $\mathcal{S}$-measurable function
  - $|f_k(x)| \leq |f_{k+1}(x)| \leq |f(x)|$ for all $k \in \mathbb{Z}^+$ and all $x \in$ X
  - $\lim_{k \longrightarrow \infty} f_k(x) = f(x)$ for every $x \in$ X
  - $f_1, f_2, \ldots$ converges uniformly on X to f if f is bounded!

Integration with respect to a Measure

Integral in terms of simple functions

Let $(X, \mathcal{S}, \mu)$ be a measure space and $f : X \longrightarrow [0, \infty]$ is $\mathcal{S}$-measurable, then

$$\int f d\mu = sup\{\int s d\mu : \text{s simple}_{0 \leq s \leq f}$$

Properties of Integration

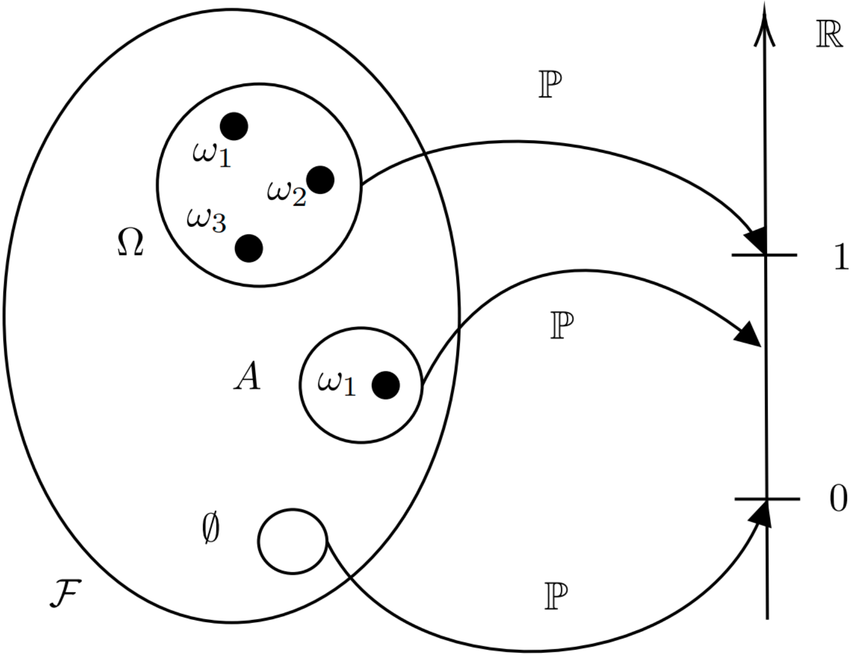
Suppose $(X, \mathcal{S}, \mu)$ is measure space and $f, g : X \longrightarrow [0, \infty]$ are $\mathcal{S}$-measurable functions

- Integration preserves order! s.t $f(x) \leq g(x)$ for all $x \in X$ then $\int f d\mu \leq \int g d\mu$
- Additivity, $\int (f + g) d\mu = \int f d\mu + \int g d\mu$
- Can break function, $X \longrightarrow [-\infty, \infty]$, into its positive and negative regions and take difference to integrate real valued functions
- Homogeneous, $\int c f d\mu = c \int f d\mu$
- Absolute Value, $|\int f d\mu| = \int |f| d\mu$

Probability Measures and Spaces

What is a probability measure?

- Suppose $\mathcal{F}$ is a σ-algebra on a set Ω, then a probability measure on $(\Omega, \mathcal{F})$ is a measure P on $(\Omega, \mathcal{F})$ such that $P(\Omega) = 1$
- Ω is called the sample space
- An event is an element of $\mathcal{F}$
- Given an event A, $P(A)$ is called the probability of A
- If P is the probability measure on $(\Omega, \mathcal{F})$ then $(\Omega, \mathcal{F}, P)$ is called the probability space



ℝ
ℙ
ω_1
ω_2
ω_3
Ω
1
ℙ
A
ω_1
0
∅
ℙ
$\mathcal{F}$

Independence and Conditional Probability

Independent Events

Suppose $(\Omega, \mathcal{F})$ then $(\Omega, \mathcal{F}, P)$ is the probability space,

- Two events, A and B, are independent if $P(A \cap B) = P(A) \cdot P(B)$
- For more than two events, $P(A_{k_1} \cup A_{k_2} \cup ...A_{k_n}) = P(A_{k_1})...P(A_{k_n})$ for $k_1, ..., k_n$

Conditional Probability

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and B us ab event with $P(B) > 0$. Can define $P_B : \mathcal{F} \longrightarrow [0, 1]$ by

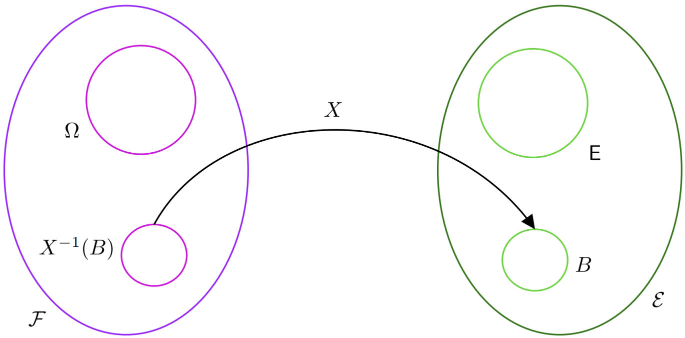
$$P_B(A) = P(A|B) = \frac{P(A \cap B)}{P(B)}$$

If $A \in \mathcal{F}$, then $P_B(A)$ is called the conditional probability of A given B.

Random Variables

How random is random?

- A random variable can be discrete or continuous, but in either case is a function that maps from $\Omega \longrightarrow \mathbb{R}$
- A variable that depends on the outcome of a random process



Probability Distribution and Distribution Function

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and X is a random variable.

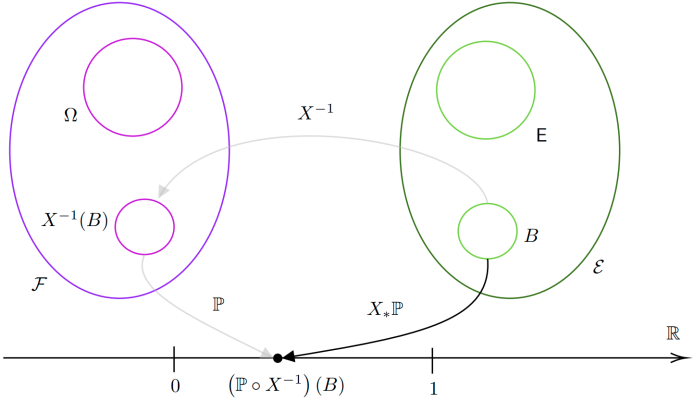
Probability Distribution

The probability distribution of X is the probability measure P_X defined on $(\mathbb{R}, \mathcal{B})$ by $P_X(B) = P(X \in B) = P(X^{-1}(B))$

Distribution Function

The distribution function of X is the function $\tilde{X} = P_X((-\infty, s]) = P(X \leq s)$

Measure to define probability



Nice Properties

- Using measure we can define expectation, independence, variance and standard deviation of random variables
- Everything we use already in probability!

Applications to Signal Filtering

- Measure is the foundation of stochastic calculus, Ito's Formula, Stratonovich's Integral, and stochastic flow
- Can create filtering equations (Zakai) based on stochastic calculus
- Kalman-Bucy filtering, i.e continuous Kalman filter!
- What are the limitations of this filtering in this way?

References

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The End