Loss Models Prelims for Actuarial Students
Wednesday, January 12, 2022, 12-4 pm
Location/modality to be announced

Instructions:

1. There are five (5) questions here and you are to answer all five. Each question is worth 20 points.
2. Hand-held calculators are permitted.
3. Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
4. Please write legibly. Points will be deducted for incoherent, incorrect, and/or irrelevant statements.

Question No. 1:

Consider a collective risk model $S = \sum_{i=1}^{N} X_i$, where X_i for $i = 1, 2, \ldots$ are i.i.d. random variables with the same distribution as X and is independent of the claim frequency N. Further assume the probabilities of X are given by

$$\Pr(X = 1) = 0.50, \qquad \Pr(X = 2) = 0.30, \qquad \Pr(X = 3) = 0.20,$$

and a zero-inflated Poisson distribution is applied to model the claim frequency N with

$$\Pr(N = n) = c\left(e^{-\lambda}\frac{\lambda^n}{n!}\right), \qquad n = 1, 2, \cdots,$$

where $\lambda = 4$ and c is a positive constant yet to be determined.

(a) It is given that $\Pr(N = 0) = 0.90$. Determine c.

(b) Calculate the mean and variance of N.

(c) Calculate the mean and variance of S.

(d) Calculate the probability $\Pr(S > \mathrm{E}[S] + \sigma[S])$, where $\sigma[S]$ is the standard deviation of S.

Question No. 2:

Individual loss amount X follows a two-parameter Weibull distribution with mean 6 and variance 180. An insurance policy on loss X has a deductible amount of 1 and a policy limit of 30 per loss.

Assume loss amount increased due to inflation by 10% uniformly.

(a) Given that $\tau = 1/2$, determine the value of the parameter θ.

(b) Calculate the expected value of claims per loss after the inflation. You may leave your answer in terms of the incomplete gamma function.

(c) Calculate the variance of claims per loss after the inflation. You may leave your answer in terms of the incomplete gamma function.

Question No. 3:

Suppose the ground-up loss X follows an exponential distribution with mean $\mathrm{E}[X] = 10$. There are two types of insurance policies available to cover such a loss:

- The first policy is a deductible insurance, where the policy payment is given by

$$I_1 = (X - d)^+, \qquad d > 0.$$

- The second policy is a coinsurance, where the policy payment is given by

$$I_2 = c\,X, \qquad 0 < c < 1.$$

(a) Let $\mathrm{E}[I_1] = \mu$, determine the deductible amount d using μ in the first policy.

(b) If both policies have the same expected payment (i.e. $\mathrm{E}[I_1] = \mathrm{E}[I_2]$), identify the relation between d and c. Please express d as a function of c.

(c) Assume $\mathrm{E}[I_1] = \mathrm{E}[I_2] = 8$, which policy is more preferable to the insured if she prefers a smaller variance in payment? Justify your answers.

Question No. 4:

Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution with mean μ and variance γ, where $\gamma > 0$. Note: We have used γ symbol for convenience; yes, there is no power for convenience.

(a) If γ is known, derive the maximum likelihood estimator of μ and derive the asymptotic distribution of this estimator.

(b) If μ is known, derive the maximum likelihood estimator of γ and derive the asymptotic distribution of this estimator.

(c) If both μ and γ are unknown parameters, derive the maximum likelihood estimators of μ and γ.

(d) Derive the joint asymptotic distribution of the estimators in (c).

Question No. 5:

Suppose you are interested in estimating the parameter $\theta = \Pr(X > b)$, for some random variable X and some constant b in the range of the distribution of X. For a single simulated value of X, the raw estimator is given by

$$I = I(X > b) = \begin{cases} 1, & \text{if } X > b \\ 0, & \text{if } X \le b \end{cases}$$

(a) Explain and justify why the two random variables I and $-X$ are negatively correlated.

(b) Because of the result in (a), a natural controlled estimator is defined by

$$Y = I - k(X - \mathrm{E}[X]).$$

In the case where X is Uniform on $(0, 1)$, determine an expression for the constant k, in terms of b, that minimizes the variance of Y. Determine the percentage of variance reduction over that of the raw estimator.

(c) Repeat (b) if X has an Exponential distribution with mean 1.

—— end of exam ——

APPENDIX

A random variable X is said to have a Gamma distribution with scale parameter $a > 0$ if its density has the form

$$f(x) = \frac{a^b x^{b-1} e^{-ax}}{\Gamma(b)}, \quad \text{for } x > 0.$$

A random variable X is said to be a two-parameter Pareto(α, θ) if its cumulative distribution function has the form

$$F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}, \quad \text{for } x > 0.$$

Its mean and variance are, respectively,

$$\mathrm{E}[X] = \frac{\theta}{\alpha - 1} \quad \text{and} \quad \mathrm{Var}[X] = \frac{\alpha\theta^2}{(\alpha-1)(\alpha-2)},$$

provided they exist.

A random variable X is said to have a two-parameter Weibull distribution if its density has the form

$$f(x) = \frac{1}{x}\tau(x/\theta)^{\tau} e^{-(x/\theta)^{\tau}}, \quad \text{for } x > 0.$$

This distribution satisfies the following:

$$\mathrm{E}[X^k] = \theta^k \Gamma(1 + k/\tau), \quad \text{for any } k > -\tau.$$

and

$$\mathrm{E}[(X \wedge x)^k] = \theta^k \Gamma(1 + k/\tau)\Gamma[1 + k/\tau; (x/\theta)^{\tau}] + x^k e^{-(x/\theta)^{\tau}}, \quad \text{for any } k > -\tau.$$