Probability Prelim Exam for Actuarial Students
January 2022

Instructions

(a). The exam is closed book and closed notes.

(b). Answers must be justified whenever possible in order to earn full credit.

(c). Points will be deducted for mathematically incorrect statements.

1. (10 points) Let X_1, X_2, ... be a sequence of independent and identically distributed nonnegative random variables. Find $a \in [-\infty, \infty]$ such that
$$\limsup_{n\to\infty} \frac{X_n}{n} = a \quad \text{a.s.}$$

2. (10 points) Let X and Y be random variables with finite variances. Suppose that $E[X|Y] = Y$ and $E[Y|X] = X$. Show that $X = Y$ a.s.

3. (10 points) Find a random variable X and a positive number α such that
$$P\left(|X - E[X]| \geq \alpha\right) = \frac{\mathrm{Var}(X)}{\alpha^2} > 0.$$

4. (10 points) For each $i = 0, 1, 2, \ldots$, let $f_i : [0,1] \to [0,10]$ be a Borel-measurable function with $\int_0^1 f_i \mathrm{d}\lambda = 1$ and μ_i a probability measure defined by
$$\mu_i(A) = \int_A f_i \mathrm{d}\lambda, \quad A \in \mathcal{B}([0,1]),$$
where λ is the Lebesgue measure. Suppose that for each $x \in [0,1]$, $\lim_{n\to\infty} f_n(x) = f_0(x)$. Show that $\{\mu_n\}$ converges weakly to μ_0.

5. (10 points) Let P and Q be two probability measures on $([0,\infty), \mathcal{B}([0,\infty)))$ that are defined as:
$$P(A) = \int_A e^{-x}\mathrm{d}x, \quad Q(A) = \int_A \frac{1}{2}e^{-x/2}\mathrm{d}x, \quad A \subseteq [0,\infty).$$
For each of the probability measures, determine whether it is absolutely continuous with respect to the other. If it is, find the Radon-Nikodym derivative explicitly.

6. (10 points) Let Z be a standard Gaussian random variable. Find t such that
$$E\left[\mathrm{Var}\left(Z|Z < t\right)\right]$$
is minimized.

7. (10 points) Let $\{X_n : n \geq 1\}$ be a sequence of independent and identically distributed random variables with the following distribution:
$$P(X_1 = 1) = \frac{2}{3}, \quad P(X_1 = -1) = \frac{1}{3}.$$
Let $S_0 = 0$, $S_n = X_1 + X_2 + \cdots + X_n$ for $n \geq 1$, and $\tau = \min\{n > 0 : S_n = 10\}$. Calculate $\mathrm{Var}(\tau)$.