Real Analysis Preliminary Exam, January 2022

Instructions and notation:

(i) Give full justifications for all answers in the exam booklet.

(ii) Lebesgue measure on $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$ and $\mathrm{d}x$ corresponds to Lebesgue integration.

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1. (10 points) Prove or disprove the following statements.
   1. If $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ then $\mathcal{L}^n(S^{n-1}) = 0$.
   2. Every nonnegative continuous $f \in L^1(\mathbb{R})$ satisfies $\limsup_{x\to+\infty} f(x) \in [0, \infty)$.
   3. If $f : (a, b) \to \mathbb{R}$ is differentiable then f' is Lebesgue measurable.

2. (10 points) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\{f_n\}_{n\in\mathbb{N}}, f$, be measurable functions. Show that if $f_n \geq 0$ for all $n \in \mathbb{N}$ and $f_n \xrightarrow{\mu} f$ then
$$\int f \,\mathrm{d}\mu \leq \liminf_{n\to+\infty} \int f_n \,\mathrm{d}\mu.$$

3. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. Show that there exists some $C > 0$ such that
$$\|f\|_1 \leq C(\|f\|_2 + \|x^2 f\|_2).$$

4. (10 points) Let $f : [0, 1] \to \mathbb{R}$ be a Borel measurable function such that $\int_0^1 |f(t)| \,\mathrm{d}t < \infty$. Prove that the function
$$h(x) = \int_x^1 t^{-1} f(t) \,\mathrm{d}t$$
is integrable on $[0, 1]$ and $\int_0^1 f(t) \,\mathrm{d}t = \int_0^1 h(t) \,\mathrm{d}t$.

5. (10 points) Find the following limit
$$\lim_{n\to\infty} \int_0^n \left(\frac{\sin x}{x}\right)^n \,\mathrm{d}x.$$

6. (10 points) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. A set $A \in \mathcal{A}$ is called an atom if $\mu(A) > 0$ and for all $B \subset A, B \in \mathcal{A}$, we have that $\mu(B) = 0$ or $\mu(B) = \mu(A)$. Show that if there exists an $\varepsilon > 0$ such that $\mu(A) \geq \varepsilon$ for all $A \in \mathcal{A}$ with $\mu(A) > 0$ then every $A \in \mathcal{A}$ with $\mu(A) > 0$ contains an atom.