Abstract Algebra Prelim Aug. 2022

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. (**10 pts**) Let R be a PID. Prove the Chinese remainder theorem in R: if a and b in R are relatively prime, $R/(ab) \cong R/(a) \times R/(b)$ as rings.

2. (**10 pts**) Let A be a commutative ring. An element a in A is called *nilpotent* when $a^n = 0$ for some $n \geq 1$. The exponent n may depend on a.

   (a) (**6 pts**) Prove the set of all nilpotent elements of A is an ideal in A.

   (b) (**4 pts**) If an integer $m \geq 2$ has distinct prime factors $p_1, \ldots, p_r$, show the nilpotent elements of $\mathbf{Z}/m\mathbf{Z}$ form the principal ideal $(p_1 p_2 \cdots p_r \bmod m)$.

3. (**10 pts**) Let $n \geq 3$ and let r and s be the standard generators of the dihedral group of order $2n$: r has order n, s has order 2, and $srs^{-1} = r^{-1}$ (equivalently, $sr = r^{-1}s$).

   (a) (**5 pts**) For an *automorphism* f of this dihedral group $\langle r, s\rangle$, show $f(r) = r^a$ for some integer a where $(a, n) = 1$ and $f(s) = r^b s$ for some integer b.

   (b) (**5 pts**) Conversely, given integers a and b such that $(a, n) = 1$, show there is a unique automorphism f of $\langle r, s\rangle$ such that $f(r) = r^a$ and $f(s) = r^b s$.

4. (**10 pts**) Prove Cauchy's theorem: if a finite group G has order divisible by a prime p, then there is an element of G with order p. (In your solution, you may use the class equation or orbit-stabilizer formula without proof, but you may not use the Sylow theorems.)

5. (**10 pts**) Let V be an finite-dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle$.

   (a) (**4 pts**) Prove each element φ in the dual space V^* has the form $\varphi(v) = \langle v, w\rangle$ for a unique w in V, where w depends on φ.

   (b) (**6 pts**) For each linear map $A\colon V \to V$ and $w \in V$, the function $v \mapsto \langle A(v), w\rangle$ is in V^*, so by (a) there is one $u \in V$, depending on A and w, such that $\langle A(v), w\rangle = \langle v, u\rangle$ for all $v \in V$. Write u as $A^*(w)$. That defines the adjoint map $A^*\colon V \to V$, so $\langle A(v), w\rangle = \langle v, A^*(w)\rangle$ for all $v, w \in V$. Prove A^* is linear.

6. (**10 pts**) Give examples as requested, with justification.

   (a) (**2.5 pts**) Generators of a 3-Sylow subgroup of the symmetric group S_6.

   (b) (**2.5 pts**) An irreducible polynomial in the polynomial ring $\mathbf{Z}[x]$ that fails the reduction mod p test when $p = 2$ (its reduction mod 2 is not irreducible mod 2).

   (c) (**2.5 pts**) A unit in $\mathbf{Z}[\sqrt{6}]$ besides ± 1.

   (d) (**2.5 pts**) A nonzero matrix in $\mathrm{M}_2(\mathbf{R})$ that is orthogonal to $\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$ with respect to the inner product $\langle A, B\rangle = \mathrm{Tr}(AB^\top)$ on $\mathrm{M}_2(\mathbf{R})$. (You don't have to check $\langle \cdot, \cdot \rangle$ is an inner product. Note the transpose on B in the inner product.)