TOPOLOGY PRELIM, AUGUST 2022

Convention

- Let E and F be two sets. We denote $E \setminus F = \{x \in E \mid x \notin F\}$.
- Unless otherwise indicated, the space $\mathbb{R}^n$ and its subsets given below are endowed with the standard topology.

1. Let Ω be an open set in $\mathbb{R}^2$ with boundary $\partial\Omega$. Let B be an open subset of Ω such that $\partial B \cap \partial\Omega$ is nonempty. Then, is it true that $\partial B \cap \partial\Omega = \overline{B} \cap (\mathbb{R}^2 \setminus \Omega)$? Here $\overline{B}$ denotes the closure of B in $\mathbb{R}^2$. Prove your assertion.

2. Let Z be a topological space and let $f : Z \to S^5$ be a continuous map. Show that, if f is not surjective, then f is null homotopic.

3. Let $Q = [0,1] \times [0,1]$, and let Γ be the subset of Q given by
$$\Gamma = (\{0\} \times [0,1]) \cup (\{1\} \times [0,1]) \cup ([0,1] \times \{0\}).$$
Prove or disprove the statement: The subspace Γ is a *deformation retraction* of Q; that is, there is a continuous map $\varphi : Q \to \Gamma$ such that $\varphi(a) = a$ for all $a \in \Gamma$ and that φ is homotopic to the identity map of Q.

4. A map $g : X \to Y$ between two topological spaces is called *proper* if for every compact subset K of Y, the preimage $g^{-1}(K)$ is compact in X. Now let X be a Hausdorff space and let $g : X \to \mathbb{R}^3$ be a proper continuous map. Show that the image $g(X)$ is closed in $\mathbb{R}^3$.

5. Let q_1 and q_2 be two distinct points in the torus $\mathbb{T}^2 = S^1 \times S^1$. Find the fundamental group $\pi_1(\mathbb{T}^2 \setminus \{q_1, q_2\})$.

6. Let $\mathcal{M}$ be the set of 2×2 real matrices with the topology obtained by regarding $\mathcal{M}$ as $\mathbb{R}^4$. Let $\mathcal{U} = \{A \in \mathcal{M} \mid \det A \neq 0\}$. Find all connected components of $\mathcal{U}$. Prove your assertion.