Real Analysis Preliminary Exam, August 2022

Instructions and notation:

(i) Give full justifications for all answers in the exam booklet.

(ii) The set difference of two sets A, B is defined as $A\Delta B := (A \setminus B) \cup (B \setminus A)$. The characteristic function of a set A is denoted by $\mathbf{1}_A$. Lebesgue measure on $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$ and $\mathrm{d}x$ corresponds to Lebesgue integration in $\mathbb{R}$.

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1. (10 points) Let $(X, \mathcal{B}, \mu)$ be a measure space. For any $A, B \in \mathcal{B}$, let

$$d(A, B) = \mu(A\Delta B) = \int |\mathbf{1}_A - \mathbf{1}_B| \,\mathrm{d}\mu.$$

   (a) (5 points) Show that

$$d(A, C) \leq d(A, B) + d(B, C)$$

   for all $A, B, C \in \mathcal{B}$.

   (b) (5 points) Let $A_1, B_1, A_2, B_2, \ldots \in \mathcal{B}$. Show that

$$d\left(\cup_{n\geq 1} A_n, \cup_{n\geq 1} B_n\right) \leq \sum_{n=1}^{\infty} d(A_n, B_n).$$

2. (10 points) Let $(X, \mathcal{F}, \mu)$ be a measure space and $g : X \to [0, \infty]$ a measurable function. Define

$$\nu(E) = \int_E g \,\mathrm{d}\mu \qquad \text{for all } E \in \mathcal{F}.$$

   Show that ν is a measure, and then show that

$$\int f \,\mathrm{d}\nu = \int fg \,\mathrm{d}\mu$$

   for any measurable function $f : X \to [0, \infty]$.

3. Compute the limit

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^{-n} \cos x \,\mathrm{d}x.$$

4. (10 points) Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$. Let $p, q \in [1, \infty]$ with $p \leq q$. Show that

$$\|f\|_p \leq \|f\|_q,$$

   for all $p, q \in [1, \infty]$ with $p \leq q$ and all functions $f \in L^q(\mu)$.

5. (10 points) Let $(X, \mathcal{F}, \mu)$ be a measure space, $(Y, \mathcal{G})$ a measurable space, and $\phi : X \to Y$ a measurable map. Define

$$\nu(A) = \mu\left(\phi^{-1}(A)\right) \qquad \text{for all } A \in \mathcal{G}.$$

   Show that ν is a measure and

$$\int f \,\mathrm{d}\nu = \int f \circ \phi \,\mathrm{d}\mu$$

   for any measurable function $f : Y \to [-\infty, \infty]$ for which the integral on the right hand side is defined.

6. (10 points) Let E and F be Borel subsets of $\mathbb{R}^2$, such that

$$\mathcal{L}^1(E_x) = \mathcal{L}^1(F_x) \qquad \text{for all } x \in \mathbb{R},$$

where $A_x = \{y \in \mathbb{R} : (x, y) \in A\}$ denotes the x-section of any $A \subset \mathbb{R}^2$. Show that $\mathcal{L}^2(E) = \mathcal{L}^2(F)$.

7. (10 points) Let $\mu_1, \mu_2, \ldots$ be a sequence of Radon measures on a locally compact Hausdorff space X. Suppose also that

$$\lim_{n\to\infty} \int f \, \mathrm{d}\mu_n$$

exists in $\mathbb{R}$ for every $f : X \to \mathbb{R}$ that is continuous of compact support. Show that there is a Radon measure μ on X such that

$$\int f \, \mathrm{d}\mu = \lim_{n\to\infty} \int f \, \mathrm{d}\mu_n \text{ for all } f \in C_c(X).$$