Complex Analysis Preliminary Exam, January 2023

Instructions and notation:

(i) Give full justifications for all solutions in the exam booklet. Four completely correct solutions will guarantee a Ph.D. pass.

(ii) Clearly state which theorems you are using and verify their assumptions.

(iii) For $p \in \mathbb{C}$ and $R > 0$, $B(p,R) = \{z \in \mathbb{C} : |z - p| < R\}$ and $\bar{B}(p,R) = \{z \in \mathbb{C} : |z - p| \leq R\}$.

1. Evaluate the following integral and justify your answer

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 1}\, \mathrm{d}x.$$

2. Let $U \subset \mathbb{C}$ be an open set such that $B(0,1) \subset U$. Show that if $f : U \to \mathbb{C}$ is holomorphic then

$$\max_{|z|=1} \left| \frac{1}{z} - f(z) \right| \geq 1.$$

3. Let f, g be entire functions such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that $f = cg$ for some constant $c \in \mathbb{C}$.

4. Let $U \subset \mathbb{C}$ be a simply connected domain and let $f : U \to \mathbb{C}$ be a holomorphic function such that $f^{-1}(\{0\})$ is finite. If every zero of f has even order show that f has a holomorphic square root in U, that is there exists a holomorphic function $g : U \to \mathbb{C}$ such that $g^2 = f$.

5. Let $f : \bar{B}(0,1) \to \mathbb{C}$ be a continuous non-constant function which is holomorphic in $B(0,1)$. Show that if $f(\partial B(0,1)) \subset \partial B(0,1)$ then $f(\bar{B}(0,1)) = \bar{B}(0,1)$.

6. Let $H := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. If $\mathsf{Aut}(H) := \{f : H \to H \text{ such that } f \text{ is biholomorphism}\}$. Show that

$$\mathsf{Aut}(H) = \left\{ f \in \text{Möb} : f(z) = \frac{az + b}{cz + d} \text{ where } a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\},$$

where Möb denotes the family of Möbius maps.

7. Suppose that $u : \mathbb{C} \to \mathbb{R}$ is harmonic and there exist constants $a, b > 0$ such that

$$u(z) \leq a \log |z| + b \quad \text{for } |z| > 1.$$

Show that u is a constant function.