GEOMETRY AND TOPOLOGY PRELIM, JANUARY 2023

1. Prove that the lower limit topology on $\mathbb{R}$, which is generated by a basis with intervals of the form $[a, b)$, is first countable but not second countable.

2. Let X and Y be topological spaces and let $\pi_X : X \times Y \to X$ be the projection map.
(a) Give an example of X and Y for which π_X is not a closed map.
(b) Prove that if Y is compact, then π_X is a closed map.

3. (a) State Urysohn's lemma.
(b) Prove that a connected, normal, and Hausdorff space with more than one element is uncountable.

4. Prove that there exists no retraction from the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ to

$$A = \left(\{x_0\} \times \mathbb{S}^1\right) \cup \left(\mathbb{S}^1 \times \{y_0\}\right),$$

where $(x_0, y_0) \in \mathbb{T}^2$.

5. Compute the fundamental group of the space X obtained from two tori $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ by identifying a circle $\mathbb{S}^1 \times \{x_0\}$ in one torus with the corresponding circle $\mathbb{S}^1 \times \{x_0\}$ in the other.

6. Let $p : (Y, y_0) \to (X, x_0)$ and $q : (Z, z_0) \to (X, x_0)$ be covering maps such that

$$p_* \left(\pi_1(Y, y_0)\right) = q_* \left(\pi_1(Z, z_0)\right).$$

Prove that there exists a homeomorphism $f : Y \to Z$ such that

$$q \circ f = p \quad \text{and} \quad f(y_0) = z_0.$$

All spaces are assumed path-connected and locally path-connected. You may state and use, without proof, the Lifting Criterion and the Unique Lifting Property.