# Real Analysis Preliminary Exam, January 2023

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*Instructions and notation:*

(i) Give full justifications for your answers in the exam booklet.

(ii) Lebesgue measure on $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$ and $\mathrm{d}x$ corresponds to Lebesgue integration in $\mathbb{R}$.

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1. (10 points) Prove or disprove three of the following statements.

   (a) If $A \subset \mathbb{R}$ and $\mathcal{L}^1(A) > 0$ then there exist $x, y \in A$ such that $x - y \notin \mathbb{Q}$.

   (b) Let $f : [0,1] \to \mathbb{R}$. If the sets $\{x \in [0,1] : f(x) = c\}$ are Lebesgue measurable for every $c \in \mathbb{R}$, then $f$ is measurable.

   (c) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $\{f_n\}_{n\in\mathbb{N}}, f : X \to \mathbb{R}$ be measurable functions. If $f_n \to f$ in measure then $f_n \to f$ almost everywhere.

   (d) If $\mu$ and $\nu$ are measures on a measurable space $(X, \mathcal{F})$ that have exactly the same sets of measure zero (that is, $\mu(A) = 0$ if and only if $\nu(A) = 0$) then $L^\infty(\mu) = L^\infty(\nu)$.

2. (10 points) Let $(X, \mathcal{A}, \nu)$ be a measure space, and let $A_1, A_2, \ldots \in \mathcal{A}$. Consider the set $A = \cap_{n\geq 1} \cup_{k\geq n} A_k$. Show that if $\sum_{n=1}^{\infty} \nu(A_n) < \infty$ then $\nu(A) = 0$.

3. (10 points) Compute the limit
$$\lim_{n\to\infty} \int_0^n \frac{\cos nx}{1 + nx^4}\,\mathrm{d}x.$$

4. (10 points) Let $\mu$ and $\nu$ be finite measures on a measurable space $(X, \mathcal{F})$.

   ( a) Show that there exists a non-negative measurable function $g$ on $X$, such that
$$\int (1-g) f \,\mathrm{d}\nu = \int g f \,\mathrm{d}\mu,$$
   for all non-negative measurable functions $f$ on $X$.

   ( b) Show, moreover, that $g$ can be chosen to satisfy $g \leq 1$ everywhere on $X$.

5. (10 points) Let $(X, \mathcal{F}, \mu)$ be a measure space, and $g$ a non-negative measurable function on $X$ such that
$$\mu\left(\{x \in X : g(x) > 0\}\right) > 0.$$
   Show that the function $\phi$ given by
$$\phi(r) = \log \int g^r \,\mathrm{d}\mu \qquad \text{for } r \in (0, \infty)$$
   is convex on the interval $\{r > 0 : \int g^r \,\mathrm{d}\mu < \infty\}$.

6. (10 points) Let $p : \mathbb{R}^n \to \mathbb{R}$ be a polynomial in $n$ variables, where $n \geq 2$. Show that the set $p^{-1}(\{0\}) \subset \mathbb{R}^n$ has Lebesgue measure 0.

7. (10 points) Let $X$ and $Y$ be compact Hausdorff spaces, $\mu$ a Radon measure on $X$ and $\nu$ a Radon measure on $Y$. Show that there is a Radon measure $\lambda$ on $X \times Y$ such that
$$\int F \,\mathrm{d}\lambda = \int_X \left[\int_Y F(x,y)\,\mathrm{d}\nu(y)\right] \mathrm{d}\mu(x)$$
   for all continuous functions $F : X \times Y \to \mathbb{R}$.