Applied Math Prelim January 2023

Student ID

1. Let T be a linear transformation between normed linear spaces X, Y.

   (a) (10 pts) Show that T is continuous iff $\sup_n ||Tx_n||_Y < \infty$ for every sequence $x_n \to 0$ in X.

   (b) (15 pts) Show that T is compact iff $Tx_n \to 0$ in Y whenever $x_n \rightharpoonup 0$ in X.

2. Let operator $A = u'' + 4u$.

   (a) (10 pts) Find the Green's function G of operator A subject to boundary conditions $u'(0) = 0$, $u(1) = 0$.

   (b) (5 pts) Find eigenvalues of A subject to the above boundary conditions.

   (c) (15 pts) Let $T : L^2(0,1) \to L^2(0,1)$ be defined by

   $$Tf(x) = \int_0^1 G(x,y)\, f(y)dy.$$

   Show that T is compact and find $\|T\|$.

3. Let X, Y be normed linear spaces, $D \subset X$ is open. $f : D \to Y$ is a mapping.

   (a) (5 pts) State the definition of Fréchet derivative of f.

   (b) (5 pts) Let $X = Y = C[0,1]$ with sup-norm. Let $v_i \in X$, $t_i \in [0,1]$ and $f(x) = \sum_{i=1}^{n} x^2(t_i)\, v_i$. Prove that f is Fréchet differentiable at all $x \in X$ and find a formula for $f'(x)$.

4. Let K be a nonempty convex set in Hilbert space X.

   (a) (5 pts) Prove the paralleogram identity $2\|x\|^2 + 2\|y\|^2 = \|x-y\|^2 + \|x+y\|^2$ for all $x, y \in X$.

   (b) (5 pts) Show that there is a unique point in K which is closest to x. Denote this point by Px.

   (c) (5 pts) Show that $\mathcal{R}\langle x - Px, v - Px\rangle \le 0$ for any $v \in K$. Here $\mathcal{R}\langle\cdot,\cdot\rangle$ stands for the real part of the inner product $\langle\cdot,\cdot\rangle$.

   (d) (5 pts) Show that $\|Px - Py\| \le \|x - y\|$ for any $x, y \in X$.

5. (15 pts) Given $U(x,y) = \ln \frac{1}{\sqrt{x^2+y^2}}$. Find ΔU in the distributional sense.