Loss Models Prelims for Actuarial Students
Wednesday, Jan 11, 2023, 12:00 - 4:00 PM
MONT 313

Instructions:

1. There are five (5) questions here and you are to answer all five. Each question is worth 20 points.
2. Hand-held calculators are permitted.
3. Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
4. Please write legibly. Points will be deducted for incoherent, incorrect, and/or irrelevant statements.

Question No. 1:

Suppose the claim severity X is given by $X = \dfrac{1}{Y}$, in which $Y \sim \mathcal{U}(0.01, 1)$, i.e., Y follows a uniform distribution over $(0.01, 1)$.

(a) Calculate the mean and variance of X: $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

Consider a collective risk model $S = \sum_{i=1}^{N} X_i$, in which $N \sim \mathcal{PN}(100)$ and X_i's are i.i.d. having the same distribution as X above.

(b) Calculate the mean and variance of S: $\mathrm{E}(S)$ and $\mathrm{Var}(S)$.

(c) Using the normal approximation, calculate the following probability:

$$\Pr(S > 1.25\,\mathrm{E}(S)).$$

Question No. 2:

Let T denote the survival time of a patient after a major surgery and assume that $T \sim \mathcal{U}(0, \theta)$, in which $\theta > 0$ is a constant parameter.

We initiated a study at time 0 and observed that 5 patients are alive at time 3. Among these 5 patients, 3 of them passed away at time 5, 6, and 8, respectively, and 2 of them survived by time 10, at which the study ended.

Calculate the maximum likelihood estimate of θ.

Question No. 3:

Individual loss amount X is being modeled as a distribution with density function given by

$$f(x) = \frac{1}{2(100)^2} x e^{-(x/200)^2}, \text{for } \ x > 0.$$

(a) Find the cumulative distribution function of X.

(b) Use the result in (a) to prove the following expression:

$$\mathrm{E}(X \wedge k) = 200\sqrt{\pi} \left[\Phi(\sqrt{2}k/200) - \frac{1}{2} \right],$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal.

(c) An insurance policy on loss X has an ordinary deductible of 100. Calculate the loss elimination ratio and interpret this value.

(d) Now, suppose that loss amount increased due to inflation by 50% uniformly. For the same insurance policy in (c), calculate the new loss elimination ratio after this inflation.

(e) Explain how inflation affected the loss elimination ratio.

Question No. 4:

For a portfolio of short-term insurance contracts, loss amount is modeled with a density function given by

$$f(x; \lambda) = \lambda^2 x e^{-\lambda x}, \quad x > 0, \ \lambda > 0.$$

A random sample of n loss amounts, $x_1, x_2, \ldots, x_n$, is used to estimate the parameter λ.

(a) Derive the maximum likelihood estimator of λ.

(b) Use moment generating functions to show that $Y = 2n\lambda\overline{X}$, where $\overline{X}$ is the sample mean of $X_1, X_2, \ldots, X_n$, has a chi-square distribution. State the degrees of freedom.

(c) Determine the Fisher information function, $I(\lambda)$, and use it to derive a formula for the asymptotic variance of $\widehat{\lambda}$.

(d) A random sample of 10 loss amounts resulted in a sample mean of $\overline{x} = 358$.

  (i) Use the results in (b) to calculate an exact 95% confidence interval of λ. For the correct degrees of freedom in (b), the 2.5-th and 97.5-th percentiles of the chi-square distribution are 24.433 and 59.342, respectively.

  (ii) Use the results in (c) to calculate an approximate (asymptotic) 95% confidence interval of λ. The 97.5-th percentile of a standard normal is 1.96.

(e) Comment briefly on the comparison of these confidence intervals.

Question No. 5:

(a) Consider two random variables X and Y with the following joint probabilities:

$$\Pr(X=0, Y=0) = \Pr(X=0, Y=3) = \Pr(X=6, Y=6) = \frac{1}{3}.$$

Let $\tilde{X}$ and $\tilde{Y}$ denote the Esscher transform of X and Y, respectively. For instance, the probability density function of $\tilde{X}$ is then given by

$$f_{\tilde{X}}(x) = \frac{e^{\rho x} f_X(x)}{\mathrm{E}[e^{\rho X}]}, \quad \rho > 0.$$

Show that $\mathrm{E}[\tilde{X}] > \mathrm{E}[\tilde{Y}]$ for the same $\rho = 0.5$.

(b) Consider two independent random variables U and V with the following distributions:

$$U = \begin{cases} 0, & \text{with probability } \frac{1}{4} \\ \frac{1}{2}, & \text{with probability } \frac{1}{2} \\ 1, & \text{with probability } \frac{1}{4} \end{cases} \quad \text{and} \quad V = \begin{cases} 0, & \text{with probability } \frac{1}{2} \\ \frac{1}{2}, & \text{with probability } \frac{3}{10} \\ 1, & \text{with probability } \frac{1}{5} \end{cases}.$$

Find a δ such that the subadditivity axiom fails for VaR, i.e.,

$$\mathrm{VaR}_\delta(U) + \mathrm{VaR}_\delta(V) < \mathrm{VaR}_\delta(U+V).$$

For the above chosen δ, calculate the corresponding TVaR and comment on whether the subadditivity holds.

—— end of exam ——

APPENDIX

A random variable X is said to have a Gamma distribution with rate parameter $a > 0$ and shape parameter $b > 0$ if its probability density function has the form

$$f(x) = \frac{1}{\Gamma(b)} a^b x^{b-1} e^{-ax}, \quad x > 0.$$

Its mean and variance are, respectively,

$$\mathrm{E}[X] = \frac{b}{a} \quad \text{and} \quad \mathrm{Var}[X] = \frac{b}{a^2}.$$

Note that when $a = 1/2$ and $b = k/2$, this results in a chi-square distribution with k degrees of freedom.

Areas under the Normal Distribution Curve: $P(Z \leq z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000