Preliminary Examination in Numerical Analysis
January, 2023

**Instructions:** Solve 6 out of 8 problems. Any results that your responses rely upon must be stated clearly.

Notation:

- The set of all real numbers is denoted by $\mathbb{R}$.
- The set of all complex numbers is denoted by $\mathbb{C}$.
- $\mathbb{P}^k[a,b]$ is the vector space of all real polynomials $p : [a,b] \to \mathbb{R}$ of degree $k$ or less.
- Bold font for column vectors, *e.g.*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

  in which case we say $\mathbf{x} \in \mathbb{R}^n$ ($\mathbf{x} \in \mathbb{C}^n$) if $x_i \in \mathbb{R}$ ($x_i \in \mathbb{C}$) for $1 \leq i \leq n$.
- The transpose of $\mathbf{x}$ is denoted by $\mathbf{x}^\top$.
- Double-bar font also for matrices, *e.g.* $\mathbb{A}$ (but not $\mathbb{R}$ or $\mathbb{C}$).
- The set of all complex matrices with $n$ rows and $m$ columns is denoted by $M_{n,m}$.

1. Let $\phi_j : [a,b] \to \mathbb{R}$, $-\infty < a < b < \infty$, for $j = 0, 1, \ldots, n$, $n \geq 1$, be polynomials satisfying

- $\phi_j(x)$ is a polynomial of order precisely $j$, for each $j = 0, 1, \ldots, n$,
- and the functions are $L^2$-orthonormal;

$$\int_a^b \phi_j(x)\phi_k(x)\,dx = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{otherwise} \end{cases}, \text{ for all } 0 \leq j \leq n \text{ and } 0 \leq k \leq n.$$

(a) Let $p_n(x) = \sum_{j=0}^n \sqrt{j} \cdot \phi_j(x)$. Find the limit

$$\lim_{n\to\infty} \frac{1}{n^2} \int_a^b |p_n(x)|^2\,dx.$$

(b) Prove that the roots $x_i$, $i = 1, 2, \ldots, n$, of $\phi_n(x)$ are real, simple, and that $a < x_i < b$.

2. Denote by $\|\cdot\|_V$ some vector norm on $\mathbb{C}^n$. Given any $\mathbb{A} \in M_{n,n}$, denote the induced matrix norm by

$$\mathrm{lub}_V(\mathbb{A}) = \max_{\|\mathbf{x}\|_V = 1} \|\mathbb{A}\mathbf{x}\|_V.$$

(a) Prove that $\mathrm{lub}_V(\cdot)$ is submultiplicative.

(b) Given any matrix norm $\|\cdot\|$ on $M_{n,n}$ that is consistent with $\|\cdot\|_V$, prove that the condition number, $\kappa(\mathbb{A})$, defined with respect to the matrix norm $\|\cdot\|$ for an invertible $\mathbb{A}$ satisfies

$$1 \leq \kappa(\mathbb{A}).$$

(c) Now let the vector norm $\|\mathbf{x}\|_V$ be the maximum size of the entries of $\mathbf{x}$. In the linear system

$\mathbb{A}\mathbf{x} = \mathbf{b}$, with $\mathbb{A}$ defined below, assume that the vector $\mathbf{b}$ may have up to a 2% relative error, as measured using the vector norm $\|\cdot\|_V$. Provide a numerical bound (with justification) for the corresponding relative error in computing the entries of $\mathbf{x}$.

$$\mathbb{A} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

3. Let $s_\Delta : [-1, 1] \to \mathbb{R}$ be a cubic spline function relative to the domain partition $\Delta \equiv \{-1, 0, 1\}$, so there is one *knot* at $x = 0$. Consider the set $V$ of all such $s_\Delta$ that satisfy the simultaneous conditions $s_\Delta(-1) = 0$, $s_\Delta(0) = 0$, $s_\Delta(1) = 0$, and $s_\Delta{}'(0) = 0$. Note that $V$ is a vector space. Derive a basis for $V$ (show the work for your derivation).

4. Given any real-valued function $f = f(x)$ that is continuous for all $x \in [0, 1]$ and differentiable at $x = 0$, let $\tilde{I}(f)$ denote the quadrature approximation

$$\tilde{I}(f) = \frac{1}{12}\left(5f(0) + f'(0) + 4f(1/2) + 3f(1)\right),$$

with associated quadrature error, say $R(f)$, defined by

$$R(f) = \tilde{I}(f) - \int_0^1 f(x)\,dx.$$

Given any $f \in \mathcal{C}^3[0, 1]$, prove that there exists some $y \in (0, 1)$ such that

$$R(f) = \frac{1}{144} f^{(3)}(y).$$

5. Denote by $\|\cdot\|$ the Euclidean vector norm on $\mathbb{C}^n$. Given any $\mathbb{A} \in M_{n,n}$, denote the induced matrix norm by

$$\|\mathbb{A}\| = \max_{\|\mathbf{x}\|=1} \|\mathbb{A}\mathbf{x}\|.$$

Next, let $\mathbb{A} \in M_{n,n}$ be a fixed Householder matrix, and assume $\mathbb{B} \in M_{n,n}$ is given such that $\|\mathbb{B} - \mathbb{A}\| = 1/3$.
(a) Prove that $\mathbb{B}$ is nonsingular.
(b) If $\mathbb{A}\mathbf{x} = \mathbb{B}\mathbf{y}$, prove that $2\|\mathbf{x} - \mathbf{y}\| \le \|\mathbf{x}\|$.

6. Write out the composite trapezoidal rule to estimate $\int_a^b f(x)\,dx$, $a < b$, using a partition $x_j = a + j \cdot h$, $j = 0, 1, \ldots, N$ and $h = (b - a)/N$. Prove that the quadrature error vanishes as $N \to \infty$, given that $f \in \mathcal{C}^1[a, b]$.

7. Given any real-valued function $f = f(x)$ that is continuous for all $x \in [-1, 1]$, let $I(f) = \int_{-1}^1 f(x)\,dx$. Define a quadrature approximation by

$$\tilde{I}(f) = w_{-1}f(-1) + w_0 f(0) + w_1 f(1).$$

Here, the weights $w_j$, $j = -1, 0, 1$, are real numbers such that $I(f) = \tilde{I}(f)$ whenever $f$ is a polynomial of degree four or less with $f(0) = f'(0)$ and $f(1) = f'(1)$.
(a) Prove that the weights $w_j$, $j = -1, 0, 1$, exist and are unique.
(b) Prove that there exists a positive constant $C > 0$ such that for all $f \in \mathcal{C}^5[-1, 1]$ with $f(0) = f'(0)$ and $f(1) = f'(1)$,

$$\tilde{I}(f) - I(f) = Cf^{(5)}(\xi),$$

for some $\xi \in (-1, 1)$, where $\xi$ may depend on $f$, but $C$ does not.

8. Given any matrix $\mathbf{S}$, denote by $\mathbf{S}^H$ the result of replacing each entry of $\mathbf{S}$ by its complex conjugate, then transposing the resulting matrix. Let $\mathbf{A}$ and $\mathbf{B}$ be matrices that satisfy $\mathbf{U}^H\mathbf{A}\mathbf{V} = \mathbf{B}$, where $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices. Denote by $\mathbf{A}^+$ and $\mathbf{B}^+$ the pseudoinverses of $\mathbf{A}$ and $\mathbf{B}$, respectively. Prove that $\mathbf{V}^H\mathbf{A}^+\mathbf{U} = \mathbf{B}^+$.