Probability Prelim Exam for Actuarial Students
January 2023

Instructions

(a). The exam is closed book and closed notes.

(b). Answers must be justified whenever possible in order to earn full credit.

(c). Points will be deducted for mathematically incorrect statements.

1. (10 points) Construct an example of two probability measures P and Q on the same measurable space $(\Omega, \mathscr{F})$ such that $P(A) = Q(A)$ for all $A \in \mathscr{F}$ with $P(A) < 0.5$ but $P(B) \neq Q(B)$ for some $B \in \mathscr{F}$.

2. (10 points) Let $(A_n : n \in \mathbb{N})$ be a sequence of independent events. For $x \in \mathbb{R}$, let

$$T_x = \left\{ \limsup_{n\to\infty} \left(\frac{1}{n} \sum_{i=1}^{n} 1_{A_i} \right) \leq x \right\}.$$

Show that $P(T_x) \in \{0, 1\}$.

3. (10 points) Suppose that $(X_n : n \in \mathbb{N})$ is a sequence of random variables on some probability space that converges in distribution.

   (a). Construct a concrete example of such a sequence not converging in probability.

   (b). Show that if the sequence converges in distribution to a constant random variable, then it also converges in probability.

4. (10 points) Let the law of X be given by $\mathcal{L}(X) = 0.5\mu_N + 0.5\delta_2$, where μ_N is the standard normal distribution and δ_2 is defined by $\delta_2(B) = 1_B(2)$, i.e., $\delta_2(B)$ equals 1 if $2 \in B$ and equals 0 otherwise. Compute $E(X)$ and $Var(X)$.

5. (10 points) Let $(X_n : n \in \mathbb{N})$ be a sequence of IID random variables satisfying

$$P(X_1 = -1) = 0.5, \quad P(X_1 = 0) = 0.25, \quad P(X_1 = 1) = 0.25.$$

For $n \in \mathbb{N}$, let $S_n = 1 + X_1 + \cdots + X_n$ and let $\tau_0 = \inf\{n \in \mathbb{N} : S_n = 0\}$. Compute $P(\tau_0 < \infty)$.

6. (10 points) Construct a continuous function $M(t) : \mathbb{R} \to \mathbb{C}$ satisfying $M(0) = 1$ and $|M(t)| \leq 1$ for all $t \in \mathbb{R}$ and which is not a characteristic function of any probability distribution.

7. (10 points) Let $(X_n : n \in \mathbb{N})$ be a non-negative martingale. Show that

$$P\left[\left(\sup_{1\leq n<\infty} X_n \right) \geq 1 \right] \leq E[X_1].$$