Abstract Algebra Prelim Jan. 2023

**Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.**

1. (**10 pts**) Let $G$ be a finite group and let $H$ and $N$ be subgroups of $G$, with $N$ a normal subgroup of $G$.

   (a) (**3 pts**) Prove that $NH = \{nh : n \in N, h \in H\}$ is a subgroup of $G$.

   (b) (**4 pts**) Prove that
   $$|NH| = \frac{|N||H|}{|N \cap H|}$$

   (c) (**3 pts**) Suppose $H$ is a Sylow $p$-subgroup of $G$ and $N$ is a $p$-subgroup of $G$. Prove that $N \subset H$. (This could be solved using previous parts or by other methods.)

2. (**10 pts**)

   (a) (**6 pts**) Prove that $\mathbf{Z}[x]/(x^2-2) \cong \mathbf{Z}[\sqrt{2}]$ as rings.

   (b) (**4 pts**) Prove that $\mathbf{Z}[x]/(x^2-1) \not\cong \mathbf{Z} \times \mathbf{Z}$ as rings. (Hint: for a ring $A$, consider $A/pA$ for a suitable prime $p$.)

3. (**10 pts**) Let $K$ be a field. Prove $K[x]$ is Euclidean with respect to the degree: for all $f(x)$ and $g(x)$ in $K[x]$ such that $g(x) \neq 0$, there exist $q(x)$ and $r(x)$ in $K[x]$ such that

   (i) $f(x) = g(x)q(x) + r(x)$ and (ii) $r(x) = 0$ or $\deg r(x) < \deg g(x)$,

   and such $q(x)$ and $r(x)$ are unique.

4. (**10 pts**) For a square matrix $A \in \mathrm{M}_n(\mathbf{R})$, where $n \geq 1$, let $v_1, \ldots, v_k \in \mathbf{R}^n$ be eigenvectors of $A$ and assume their associated eigenvalues $\lambda_1, \ldots, \lambda_k$ are distinct: $\lambda_i \neq \lambda_j$ for $i \neq j$.

   (a) (**5 pts**) Prove that the set $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

   (b) (**5 pts**) Further assume that $A$ is symmetric. Prove that the set $\{v_1, v_2, \ldots, v_k\}$ is orthogonal with respect to the standard inner product on $\mathbf{R}^n$.

5. (**10 pts**) Let $V$ be an $n$-dimensional vector space over a field $K$, where $n \in \mathbf{Z}^+$.

   (a) (**3 pts**) If $\varphi\colon V \to K$ is a nonzero element of the dual space of $V$, then prove $\ker\varphi$ has dimension $n-1$.

   (b) (**7 pts**) Prove every subspace $W$ of $V$ with dimension $n-1$ has the form $\ker\varphi$ for some nonzero $\varphi$ in the dual space of $V$.

6. (**10 pts**) Give examples as requested, with justification.

   (a) (**2.5 pts**) A nonabelian semidirect product $H \rtimes K$ where $H$ and $K$ are nontrivial (write the group law for your $H \rtimes K$ explicitly).

   (b) (**2.5 pts**) A group action with more than one orbit.

   (c) (**2.5 pts**) Two non-isomorphic *abelian* groups of order 8.

   (d) (**2.5 pts**) A ring $A$ where the only nilpotent element is 0 and $A$ is *not* an integral domain.