Abstract Algebra Prelim Aug. 2023

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. (**10 pts**) The prime factorization of 2023 is $7 \cdot 17^2$.
   (a) (**5 pts**) For each prime p, prove all groups of order p^2 are abelian.
   (b) (**5 pts**) Prove every group of order 2023 is isomorphic to a direct product $H \times K$, where $|H| = 7$ and $|K| = 17^2$. (That shows, by (a), that all groups of order 2023 are abelian. You can't assume in (b) that the group is abelian.)

2. (**10 pts**) Prove that if G is a finite group whose only automorphism is the identity map, then G is trivial or of order 2.

3. (**10 pts**)
   (a) (**3 pts**) State Zorn's lemma.
   (b) (**7 pts**) Use Zorn's lemma to prove every nonzero commutative ring contains a maximal ideal.

4. (**10 pts**) Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be vectors in $\mathbf{Q}^n$, where $1 \leq m \leq n$, and let $V \subseteq \mathbf{Q}^n$ be their span over $\mathbf{Q}$. Let $V_{\mathbf{R}}$ be their span over $\mathbf{R}$ in $\mathbf{R}^n$. Prove that $V_{\mathbf{R}} \cap \mathbf{Q}^n = V$.

5. (**10 pts**) Let V be an n-dimensional real vector space, where $n \geq 1$, with basis $e_1, \ldots, e_n$. Let $\langle \cdot, \cdot \rangle$ be an inner product on V and D be the $n \times n$ matrix with (i, j)-entry $D_{ij} = \langle e_i, e_j \rangle$.
   (a) (**2 pts**) For v and w in V, prove
   $$\langle v, w \rangle = [v]^\top D[w],$$
   where $[v]$ and $[w]$ are the representations of v and w as column vectors in the basis $e_1, \ldots, e_n$.
   (b) (**4 pts**) Prove that D is symmetric and use the spectral theorem to show $\det(D) > 0$. In particular D is invertible.
   (c) (**4 pts**) Assuming $e_1, \ldots, e_n$ is orthonormal for $\langle \cdot, \cdot \rangle$, show a linear map $L : V \to V$ is self-adjoint for $\langle \cdot, \cdot \rangle$ if and only if the matrix $[L]$ is symmetric, where $[L]$ is the matrix of L with respect to $e_1, \ldots, e_n$. **Remark**: You do not need (b) to solve (c).

6. (**10 pts**) Give examples as requested, with justification.
   (a) (**2.5 pts**) An infinite abelian group in which every element has finite order.
   (b) (**2.5 pts**) An irreducible polynomial of degree 3 in $\mathbf{F}_2[x]$.
   (c) (**2.5 pts**) A ring R and ideals I and J in R such that $IJ \neq I \cap J$.
   (d) (**2.5 pts**) A basis of the vector space of 2×2 real matrices with trace 0.