Applied Math Prelim August 2023

1. (16 pts) Prove that a linear transformation between two normed linear spaces is continuous iff it transforms every sequence converging to 0 into a bounded sequence.

2. (16 pts)Let $L : X \twoheadrightarrow Y$ be linear, i.e. L is a linear surjective map. Then L is interior iff there exists a constant c such that for any $y \in Y$, there exists $x \in X$ such that $Lx = y$ and $\|x\| \leq c\|y\|$.

3. (16 pts) Let Y be a linear subspace in a normed linear space X. For any $x \in X$, prove that

$$\text{dist}(x, Y) = \sup\{\phi(x) : \phi \in X^*, \phi \perp Y, \|\phi\| = 1\},$$

Here $\phi \perp Y$ means $\phi(y) = 0$ for all $y \in Y$.

4. Let $\{e_n\}$ be an orthonormal sequence in a (complex) Hilbert space X, $\{\lambda_n\}$ be a bounded sequence of complex numbers. Define operator A by $Ax = \sum_n \lambda_n \langle x, e_n \rangle \, e_n$. Show that

   (a) (16 pts) A is a compact operator iff $\lambda_n \to 0$.

   (b) (4 pts) A is Hermitian iff λ_n are all real.

5. (16 pts) Find a fundamental solution of operator A defined by $A\phi = \phi'' + 2\phi' + \phi$. Note that the answer is not unique.

6. (16 pts) If A is a compact operator defined on a normed linear space to itself. Prove that for some natural number n, the ranges of $(I + A)^n, (I + A)^{n+1}, \cdot$ are all identical.