Instructions, conventions and notation.

- Four fully justified and completely solved problems guarantee a Ph.D. pass.
- Clearly state which theorems you are using and verify their assumptions.
- Denote by $\mathbb{C}$ the complex plane and $D = \{ z \in \mathbb{C} : |z| < 1 \}$ the open unit disk.
- A region means a nonempty connected open set.
- The terminology analytic function and holomorphic function are used interchangeably.
- Entire functions are holomorphic in $\mathbb{C}$.

Problem 1. (a) Let $f(w) := \int_{w+\mathbb{R}} \frac{dz}{z^3 - z^2 + z - 1}$, where $w \in \mathbb{C}$, and $w + \mathbb{R}$ denotes the horizontal line in $\mathbb{C}$ that contains $w$ and is oriented from left to right. Find all $w$ for which $f(w)$ is defined and compute it.

(b) Let $g(w) := \int_{w+i\mathbb{R}} \frac{dz}{z^3 - z^2 + z - 1}$, where $w \in \mathbb{C}$, and $w + i\mathbb{R}$ denotes the vertical line in $\mathbb{C}$ that contains $w$ and is oriented upwards. Find all $w$ for which $g(w)$ is defined and compute it.

Problem 2. If $\alpha \in D$, prove that $F(z) := (\bar{\alpha} - z)/(1 - \alpha z)$ maps $D$ to $D$ bijectively.

Problem 3. Describe a holomorphic isomorphism between regions $A = \{ z : 0 < \text{Re}(z) < 1 \}$ and $B = \{ z : \text{Re}(z) > 0, \text{Im}(z) > 0 \}$.

Problem 4. If $\mathcal{F} := \{ f : D \to D, \text{ } f \text{ is holomorphic in } D \}$, find $\sup_{f \in \mathcal{F}} |f''(0)|$.

Problem 5. If $\mathcal{F}_0 := \{ f : D \to D, \text{ } f \text{ is holomorphic and has at least two zeros in } D \}$, is $\mathcal{F}_0$ a normal family? Give a proof of your assertion.

Problem 6. Let $u : \mathbb{C} \to \mathbb{R}$ and $v : \mathbb{C} \to \mathbb{R}$ be harmonic conjugate functions. If $u$ is bounded, is it true that $v$ is also bounded? Either provide a proof or give a counterexample.

Problem 7. If $f$ is entire and $\gamma$ is a Jordan curve satisfying $f(z) \notin \gamma$ for all $z \in \mathbb{C}$, prove that $f$ is constant.