COMPLEX ANALYSIS PRELIM

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Instructions, conventions and notation.

- Four fully justified and completely solved problems guarantee a Ph.D. pass.
- Clearly state which theorems you are using and verify their assumptions.
- Denote by $\mathbb{C}$ the complex plane and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk.
- A *region* means a nonempty connected open set.
- The terminology *analytic* function and *holomorphic* function are used interchangeably.
- Entire functions are holomorphic in $\mathbb{C}$.

Problem 1. (a) Let $f(w) := \displaystyle\int_{w+\mathbb{R}} \frac{dz}{z^3 - z^2 + z - 1}$, where $w \in \mathbb{C}$, and $w + \mathbb{R}$ denotes the horizontal line in $\mathbb{C}$ that contains w and is oriented from left to right. Find all w for which $f(w)$ is defined and compute it.

(b) Let $g(w) := \displaystyle\int_{w+i\mathbb{R}} \frac{dz}{z^3 - z^2 + z - 1}$, where $w \in \mathbb{C}$, and $w + i\mathbb{R}$ denotes the vertical line in $\mathbb{C}$ that contains w and is oriented upwards. Find all w for which $g(w)$ is defined and compute it.

Problem 2. If $\alpha \in \mathbb{D}$, prove that $F(z) := (\overline{\alpha} - z)/(1 - \alpha z)$ maps $\mathbb{D}$ to $\mathbb{D}$ bijectively.

Problem 3. Describe a holomorphic isomorphism between regions $A = \{z : 0 < \mathrm{Re}(z) < 1\}$ and $B = \{z : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$.

Problem 4. If $\mathcal{F} := \{f : \mathbb{D} \to \mathbb{D},\ f \text{ is holomorphic in } \mathbb{D}\}$, find $\sup_{f \in \mathcal{F}} |f''(0)|$.

Problem 5. If $\mathcal{F}_0 := \{f : \mathbb{D} \to \mathbb{D},\ f \text{ is holomorphic and has at least two zeros in } \mathbb{D}\}$, is $\mathcal{F}_0$ a normal family? Give a proof of your assertion.

Problem 6. Let $u : \mathbb{C} \to \mathbb{R}$ and $v : \mathbb{C} \to \mathbb{R}$ be harmonic conjugate functions. If u is bounded, is it true that v is also bounded? Either provide a proof or give a counterexample.

Problem 7. If f is entire and γ is a Jordan curve satisfying $f(z) \notin \gamma$ for all $z \in \mathbb{C}$, prove that f is constant.