GEOMETRY AND TOPOLOGY PRELIM, AUGUST 2023

1. Prove that a topological space X is Hausdorff if and only if, for each $x \in X$,

$$\bigcap_{U \text{ neighborhood of } x} \bar{U} = \{x\}.$$

Here $\bar{U}$ denotes the closure of the set U.

2. (a) Let X be a Hausdorff topological space and A and B two compact subsets of X. Prove that $A \cap B$ is compact as well.

(b) Let $X = \mathbb{N} \cup \{-\infty, \infty\}$ and define that U is open if and only if either $U \subset \mathbb{N}$, or $X \setminus U$ is finite. Find two compact sets A and B in X so that $A \cap B$ is not compact.

3. Let A and B be subsets of a topological space X so that $A \cup B$ and $A \cap B$ are connected. Prove that if A and B are closed, then both A and B are connected.

4. Compute the fundamental group of the following spaces:

(a) $X = \mathbb{R}^3 \setminus \{L\}$, where L is a line.

(b) $Y = \mathbb{R}^3 \setminus \{L_1 \cup L_2\}$, where L_1 and L_2 are two lines that do not intersect.

5. Let X and Y be path connected, locally path-connected and semi-locally simply connected spaces. Let $p_X : \widetilde{X} \to X$ and $p_Y : \widetilde{Y} \to Y$ be the universal covers.

(a) Prove that for every continuous map $f : X \to Y$ there exists a continuous map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ such that $p_Y \circ \widetilde{f} = f \circ p_X$.

(b) Find such an $\tilde{f} : \mathbb{R} \to \mathbb{R}^2$ for $f : \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ defined by $f(z) = (z^n, z^m)$, where $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, and $n, m \in \mathbb{Z}$.

6. Assume that the continuous map $f : \mathbb{S}^1 \to \mathbb{S}^1$ is nullhomotopic. Prove that there exists $x \in \mathbb{S}^1$ such that $f(x) = x$.