Probability Prelim Exam for Actuarial Students
August 2023

Instructions

(a). The exam is closed book and closed notes.

(b). Answers must be justified whenever possible in order to earn full credit.

(c). Points will be deducted for mathematically incorrect statements.

1. (10 points) Let X and Y be two random variables that are defined jointly on some probability space $(\Omega, \mathscr{F}, P)$. Suppose that $P(X \le x, Y \le y) = P(X \le x)P(Y \le y)$ for all $x, y \in \mathbb{R}$. Show that for all open intervals I, J,

$$P(X \in I, Y \in J) = P(X \in I)P(Y \in J).$$

2. (10 points) Let X be a standard normal random variable, i.e.,

$$P(X \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u.$$

Let $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$. Calculate the correlation between X^+ and X^-.

3. (10 points) Let X be a random variable such that $E[X] = 0$ and $\mathrm{Var}(X) < \infty$. Show that for all $\alpha > 0$,

$$P(X \ge \alpha) \le \frac{\mathrm{Var}(X)}{\mathrm{Var}(X) + \alpha^2}.$$

4. (10 points) Let $\{X_n\}_{n\ge 1}$ be a sequence of independent and identically distributed random variables that follow the exponential distribution with parameter 1, i.e., $P(X_1 \le x) = 1 - e^{-x}$. For $n \ge 1$, let $M_n = \max(X_1, X_2, \ldots, X_n)$. Show that $M_n - \ln n$ converges to a distribution and find the cumulative distribution function.

5. (10 points) Let μ be the geometric distribution with parameter $\frac{1}{2}$, i.e., $\mu(k) = 2^{-k}$ for $k = 1, 2, \ldots$. Let ν be the Poisson distribution with parameter 1, i.e., $\nu(k) = \dfrac{\exp(-1)}{k!}$ for $k = 0, 1, \ldots$. Show that μ is absolutely continuous with respect to ν and find the Radon-Nikodym derivative $\dfrac{d\mu}{d\nu}$.

6. (10 points) Let X and Y be discrete random variables. Let $q(x, y) = P(X = x, Y = y)$. Show that

$$E[X|Y] = \frac{\sum_x x q(x, Y)}{\sum_x q(x, Y)}, \quad a.s.$$

7. (10 points) Let $\{B_t\}_{t\ge 0}$ be a standard Brownian motion. Compute

$$E\left[(B_2 + B_3 + 1)^2 | B_1\right].$$