Real Analysis Preliminary Exam, August 2023

Instructions and notation:

(i) You may use standard results from 5111 (although none of the problems should be interpreted as a "standard result").

(ii) Try to give full justifications for your answers in the exam booklet. If your proof has a gap, mention it and specify what would be needed to fill in the gap.

(iii) We let m denote the standard Lebesgue measure on $\mathbb{R}^n$.

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1. (10 points) Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to [0, \infty)$ be a measurable function. Define the set function

$$\nu(A) = \int_A f \, \mathrm{d}\mu, \quad A \in \mathcal{A}.$$

Prove that ν is a measure and that $\int g \, \mathrm{d}\nu = \int gf \, \mathrm{d}\mu$ for all measurable $g : X \to [0, \infty)$.

2. (10 points) Let $(X, \mathcal{A}, \mu)$ be a *finite* measure space. Prove that $f \in L^1(\mu)$ if and only if

$$\sum_{k=1}^{\infty} 2^k \mu(\{x \in X : |f(x)| \geq 2^k\}) < \infty.$$

3. Let $1 < p_1 < p_2 < \infty$. You may assume m is Lebesgue measure on $\mathbb{R}$.
   (a) (5 points) Let $f_i \in L^{p_i}(m), i = 1, 2$ be nonnegative functions. Find $r = r(p_1, p_2)$ for which $(f_1 f_2)^r \in L^1(m)$.
   (b) (5 points) If $s \neq r(p_1, p_2)$, show that there exist nonnegative $f_i \in L^{p_i}(m), i = 1, 2,$ for which $(f_1 f_2)^s \notin L^1(m)$.

4. (10 points) Let

$$f_n(x) = \frac{1}{1 + x^{\frac{\sqrt{n}}{\log(n+2023)}}}, x \geq 0, n \in \mathbb{N}.$$

Find $\lim_{n\to\infty} \int_0^\infty f_n \, \mathrm{d}m$.

5. Let $T = \{(x, y) \in \mathbb{R}^2 : 0 \leq |x| \leq y \leq 1\}$, and μ be the restriction of m to T. Let $f \in L^2(T, \mu)$. Prove that
   (a) (4 points) $f \in L^1(T, \mu)$,
   (b) (6 points) $\liminf_{y\to 0^+} \int_{-y}^{y} |f(x, y)| \, \mathrm{d}x = 0$.

6. (10 points) Let $A \subset \mathbb{R}$ be Lebesgue measurable with finite Lebesgue measure. Prove that

$$\lim_{|x|\to 0} m(A \cap (x + A)) = m(A).$$

Here, $x + A = \{x + y : y \in A\}$.