Abstract Algebra Prelim Jan. 2024

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. (**10 pts**) Let $U(n) = (\mathbf{Z}/n\mathbf{Z})^\times$ be the multiplicative group of units modulo n.

   (a) (**6 pts**) For relatively prime $m \geq 2$ and $n \geq 2$, show $U(mn)$ and $U(m) \times U(n)$ are isomorphic groups by by writing down a map and proving it is an isomorphism.

   (b) (**4 pts**) Use the prime factorization $2024 = 8 \cdot 11 \cdot 23$ to compute the order of 3 in $U(2024)$. Part (a) might be helpful.

2. (**10 pts**)

   We want to define an action of the group $\mathrm{GL}_2(\mathbf{R})$ on the set $\mathbf{C} - \mathbf{R}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d},$$

   where $z \in \mathbf{C} - \mathbf{R}$. For example, $\left(\begin{smallmatrix} 2 & 3 \\ 0 & 1 \end{smallmatrix}\right)z = 2z+3$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)z = \frac{1}{z}$.

   (a) (**5 pts**) Prove the above formula defines an action of $\mathrm{GL}_2(\mathbf{R})$ on $\mathbf{C} - \mathbf{R}$. (This includes checking its values are in $\mathbf{C} - \mathbf{R}$.)

   (b) (**3 pts**) Prove the stabilizer of i for this action is $\{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) : a, b \in \mathbf{R} \text{ and } (a,b) \neq (0,0)\}$.

   (c) (**2 pts**) Prove this group action has a single orbit.

3. (**10 pts**)

   (a) (**3 pts**) Define what it means for a nonzero element of an integral domain to be irreducible.

   (b) (**7 pts**) Prove the reduction mod p test in $\mathbf{Z}[x]$: if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is monic and nonconstant and there is a prime p such that $f(x) \bmod p$ is irreducible in $(\mathbf{Z}/p\mathbf{Z})[x]$, then $f(x)$ is irreducible in $\mathbf{Z}[x]$.

4. (**10 pts**)

   (a) (**5 pts**) Prove $\mathbf{Z}[i]$ is Euclidean with respect to the norm on $\mathbf{Z}[i]$: if α and β are in $\mathbf{Z}[i]$ and $\beta \neq 0$, then there are γ and ρ in $\mathbf{Z}[i]$ such that (i) $\alpha = \beta\gamma + \rho$ and (ii) $\mathrm{N}(\rho) < \mathrm{N}(\beta)$.

   (b) (**5 pts**) Prove each ideal in a Euclidean domain is principal. (This does not use (a).)

5. (**10 pts**) Let V be a real vector space with an inner product $\langle \cdot, \cdot \rangle$ and finite dimension $n \geq 1$.

   (a) (**4 pts**) Show every (nonempty) orthogonal set of nonzero vectors in V is a linearly independent set in V.

   (b) (**3 pts**) When $w \neq 0$ in V, show $V = \mathbf{R}w \oplus U$ where $U = \{v \in V : \langle v, w \rangle = 0\}$.

   (c) (**3 pts**) Use (b) to prove every nonzero finite-dimensional inner product space has an orthogonal basis. Part (a) might be helpful here too.

6. (**10 pts**) Give examples as requested, with justification.

   (a) (**2.5 pts**) A permutation $\sigma \in S_7$ such that $\sigma(123)(4567)\sigma^{-1} = (354)(1762)$.

   (b) (**2.5 pts**) An odd prime number p such that the ideal (p) in $\mathbf{Z}[i]$ is not a prime ideal.

   (c) (**2.5 pts**) A UFD that is not a PID.

   (d) (**2.5 pts**) For $V = \mathbf{R}^2$, a linear map $L\colon V \to V$ that is self-adjoint with respect to the usual inner product on V and is not multiplication by a scalar.