COMPLEX ANALYSIS PRELIM

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Instructions, conventions and notation.

- Four fully justified and completely solved problems guarantee a Ph.D. pass.
- Clearly state which theorems you are using and verify their assumptions.
- Denote by $\mathbb{C}$ the complex plane, by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk, and by $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the upper half plane.
- A *region* means a nonempty connected open set.
- The terminology *analytic* function and *holomorphic* function are used interchangeably.
- Entire functions are holomorphic in $\mathbb{C}$.

Problem 1. Find $\displaystyle\int_0^\infty \frac{dx}{1+x^3}$ and justify your answer.

Problem 2. Find all holomorphic functions satisfying $|f(z)| \leqslant \sqrt{|z|}$ for all $z \in \mathbb{C}$ and justify your answer.

Problem 3. Describe a holomorphic isomorphism between $\mathbb{H}$ and the region $S = \{z = x + \boldsymbol{i}y \in \mathbb{C} : -x < y < x, |z| < 2\}$.

Problem 4. Let $\mathcal{F} := \{f : \mathbb{H} \to \mathbb{D},\ f \text{ is holomorphic in } \mathbb{H}\}$.

(a) Find $\sup_{f \in \mathcal{F}} |f'(\boldsymbol{i})|$.

(b) Is this supremum achieved? If yes, find all functions that achieve this supremum and justify your answer. If this supremum is not achieved, provide a proof of this assertion.

Problem 5. Let A be the ring $A := \{z \in \mathbb{C} : 1 < |z| < 2\}$. How many zeros of the function $f(z) = z^5 + 5z^2 + e^z$ belong to A? Justify your answer.

Problem 6. Let A be the same ring as in Problem 5, and let B be the punctured unit disc $B := \{z \in \mathbb{C} : 0 < |z| < 1\}$. Prove that

(a) there is a holomorphic map from $\mathbb{D}$ onto A;

(b) there is a holomorphic map from $\mathbb{D}$ onto B;

(c) A and B are not holomorphically equivalent.

Problem 7. Let $u : \overline{\mathbb{D}} \to \mathbb{R}$ be a function continuous in the closed unit disc $\overline{\mathbb{D}}$. If u is harmonic in $\mathbb{D}$ and v is its harmonic conjugate in $\mathbb{D}$, prove that v has a continuous extension to $\overline{\mathbb{D}}$.