GEOMETRY AND TOPOLOGY PRELIM, JANUARY 2024

1. Let X be a Hausdorff space and $A \subset X$ a closed subset.
(a) Prove that the quotient map $\pi : X \to X/A$ is a closed map.
(b) Prove that if X is normal, then the quotient space X/A is normal.

Recall that Z is called normal if for any disjoint closed sets $C, D \subset Z$, there exist disjoint open sets $U, V \subset Z$ with $C \subset U$ and $D \subset V$.

2. Let X and Y be topological spaces and consider the product space $X \times Y$. Let A be a compact subset of X and B be a compact subset of Y. Prove that if W is an open set containing $A \times B$, then there exists U open in X and V open in Y such that $A \times B \subset U \times V \subset W$.

3. Let X be a connected space and $f, g : X \to [0, 1]$ continuous functions with f surjective. Prove there exists $x \in X$ such that $f(x) = g(x)$.

4. Let X be a topological space and $A \subset X$ a deformation retract of X. Prove that if A is path connected, then X is path connected as well.

5. Compute the fundamental group of X, where:
(a) X is the complement of n distinct points on the cylinder $\mathbb{R} \times \mathbb{S}^1$.
(b) X is the complement in $\mathbb{R}^3$ of n distinct lines through the origin.

6. Let A be a path connected subset of a path connected space X and let $i : A \to X$ be the inclusion map of A into X. Assume X admits universal cover and let $p : \tilde{X} \to X$ be the universal covering map.

Prove that $p^{-1}(A)$ is path connected if and only if $i_* : \pi_1(A, a) \to \pi_1(X, a)$ is surjective, for any $a \in A$.