Loss Models Prelims for Actuarial Students
Friday, Jan. 12, 2023, 9:00 AM - 1:00 PM

**Instructions:**

1. There are five (5) equally-weighted questions and you are to answer all five.

2. Hand-held calculators are permitted.

3. Please provide details of your workings in the appropriate spaces provided; partial points will be granted.

4. Please write legibly. Points will be deducted for incoherent, incorrect, and/or irrelevant statements.

**Question No. 1:**

Suppose $X$ refers to the ground-up loss random variable for an insurance policy with an ordinary deductible of $d > 0$.

(a) Define Loss Elimination Ratio (LER).

(b) Prove that for an ordinary deductible of $d > 0$, $\mathbb{E}[(X - d)_+] = \mathbb{E}[X] - \mathbb{E}[X \wedge d]$, and deduce that $\text{LER} = \frac{\mathbb{E}[X \wedge d]}{\mathbb{E}[X]}$.

(c) Suppose in the subsequent year, ground-up loss increases uniformly with inflation $r$ so that $(1 + r)X$ is the ground-up loss in the subsequent year. The deductible amount stays the same. Show that the new LER is

$$\text{LER}(r) = \frac{\mathbb{E}[X \wedge \frac{d}{1+r}]}{\mathbb{E}[X]},$$

as a function of $r$.

(d) Prove that LER decreases with $r$, i.e., whenever $r_1 \geq r_2$, we have $\text{LER}(r_1) \leq \text{LER}(r_2)$.

(e) Explain intuitively (in words) why LER decreases with inflation.

**Question No. 2:**

Consider the collective risk model $S = \sum_{i=1}^{N} X_i$, in which the primary and secondary distributions are given, respectively, by

<table>
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<th>$n$</th>
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<tr>
<td>$\text{Pr}(N = n)$</td>
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and

<table>
<thead>
<tr>
<th>$x$</th>
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<th>2</th>
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<tbody>
<tr>
<td>$\text{Pr}(X = x)$</td>
<td>0.6</td>
<td>0.4</td>
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(a) Determine $\text{VaR}_\alpha(S) := \inf\{ s \in \mathbb{R} : \text{Pr}(S \leq s) \geq \alpha \}$ for all $\alpha \in (0, 1)$.

(b) Calculate $\text{CVaR}_{0.80}(S) := \frac{1}{1 - 0.80} \int_{0.80}^{1} \text{VaR}_\alpha(S) \, d\alpha$. 
(c) Now assume that the insurer wants to lower the CVaR$_{0.80}$ of the aggregate claims to 2.5, and to achieve that goal, it introduces a policy limit $u$ per claim. Denote the aggregate claims after the coverage modification by $\tilde{S}$. Note that

$$\tilde{S} = \sum_{i=1}^{N} (X_i \wedge u).$$

Calculate the desired $u$ such that CVaR$_{0.80}(\tilde{S}) = 2.5$.

**Question No. 3:**

Consider a full insurance policy (i.e., it has a zero deductible and no policy limit) and assume its aggregate claim $S$ for the current year is given by the following collective risk model:

$$S = \sum_{i=1}^{N} X_i,$$

for which $N$ follows a Poisson distribution with mean 2, and $X_i \overset{d}{=} X$ follows an exponential distribution with mean 50.

(a) Suppose the premium $P$ of such a policy is determined by the expected-value premium principle with 20% loading, i.e.,

$$P = 1.2 \times \mathbb{E}[S].$$

Calculate the premium $P$.

(b) Apply normal approximation to calculate the probability that $S > 2\mathbb{E}[S]$.

(c) Due to inflation, the claim severity in the next year is estimated to increase by 10%, but the claim frequency is not impacted by inflation. In order to keep the premium $P$ unchanged for the next year, the insurer is planning to introduce an ordinary deductible $d$.

Calculate $d$.

**Question No. 4:**

The probability mass function (pmf) of a zero-truncated Poisson random variable $X$ can be expressed as:

$$p_k = \frac{e^{-\lambda} \cdot \lambda^k}{1 - e^{-\lambda} \cdot k!}, \quad k = 1, 2, \ldots, \lambda > 0.$$

(a) Show that the first two moments of a zero-truncated Poisson distribution are:

$$\mathbb{E}[X] = \frac{\lambda}{1 - e^{-\lambda}} \quad \text{and} \quad \mathbb{E}[X^2] = \frac{\lambda + \lambda^2}{1 - e^{-\lambda}}.$$

(b) Establish the following relationship of the mean and variance as

$$\text{Var}[X] = \mathbb{E}[X](1 + \lambda - \mathbb{E}[X]),$$

and therefore, deduce that

$$\text{Var}[X] \leq \mathbb{E}[X].$$
Question No. 5:

(a) The following observations sampled from a Weibull distribution $W(\alpha, \lambda)$ are given:

54, 70, 75, 81, 84, 88, 97, 105, 109, 114, 122, 125, 128, 139, 146, 153.

Apply the percentile matching method at 20th and 70th percentiles to estimate $\alpha$ and $\lambda$.

Note: The cumulative distribution function (cdf) of $W(\alpha, \lambda)$ is

$$F_X(x) = 1 - \exp\left(-\frac{x}{\lambda}\right)^\alpha, \quad x > 0.$$ 

(b) The following claim payment amounts from an insurance policy with a deductible of 100 are recorded:

20, 80, 100, 170, 200, 900, 2400.

Assume the ground-up loss $X$ is modeled by $\mathcal{P}(\alpha, 400)$, i.e., its pdf is given by

$$f_X(x) = \frac{\alpha 400^\alpha}{(x + 400)^{\alpha+1}}, \quad x > 0.$$ 

Compute the maximum likelihood estimate (MLE) of $\alpha$.

Hint: To illustrate, for a ground-up loss of 150, the insurer pays 50 to settle this claim.
APPENDIX

A random variable $X$ is said to have Pareto distribution with parameters $\alpha > 0$ and $\theta > 0$, sometimes denoted by $\mathcal{P}(\alpha, \theta)$, if its cdf is expressed as

$$F(x) = 1 - \left( \frac{\theta}{x + \theta} \right)^\alpha.$$ 

A discrete random variable $N$ is said to belong to the $(a, b, 0)$ class of distributions if it satisfies the relation

$$\Pr(N = k) = p_k = \left( a + \frac{b}{k} \right) \cdot p_{k-1}, \quad \text{for } k = 1, 2, \ldots,$$

for some constants $a$ and $b$. Alternatively, this relation can be expressed as a linear function as

$$k \cdot \frac{p_k}{p_{k-1}} = b + ak, \quad \text{for } k = 1, 2, \ldots.$$

The initial value $p_0$ is determined so that $\sum_{k=0}^{\infty} p_k = 1$.

The Poisson distribution belongs to the $(a, b, 0)$ class of distributions with $a = 0$ and $b = \lambda$. Its mean and variance are equal:

$$\mathbb{E}[X] = \text{Var}[X] = \lambda.$$