**Loss Models Prelims for Actuarial Students**
**Friday, Jan. 12, 2023, 9:00 AM - 1:00 PM**

**Instructions:**

1. There are five (5) equally-weighted questions and you are to answer all five.
2. Hand-held calculators are permitted.
3. Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
4. Please write legibly. Points will be deducted for incoherent, incorrect, and/or irrelevant statements.

## Question No. 1:

Suppose $X$ refers to the ground-up loss random variable for an insurance policy with an ordinary deductible of $d > 0$.

(a) Define Loss Elimination Ratio (LER).

(b) Prove that for an ordinary deductible of $d > 0$, $\mathbb{E}[(X-d)_+] = \mathbb{E}[X] - \mathbb{E}[X \wedge d]$, and deduce that LER is $\text{LER} = \dfrac{\mathbb{E}[X \wedge d]}{\mathbb{E}[X]}$.

(c) Suppose in the subsequent year, ground-up loss increases uniformly with inflation $r$ so that $(1+r)X$ is the ground-up loss in the subsequent year. The deductible amount stays the same. Show that the new LER is

$$\text{LER}(r) = \frac{\mathbb{E}\left[X \wedge \frac{d}{1+r}\right]}{\mathbb{E}[X]},$$

as a function of $r$.

(d) Prove that LER decreases with $r$, i.e., whenever $r_1 \geq r_2$, we have $\text{LER}(r_1) \leq \text{LER}(r_2)$.

(e) Explain intuitively (in words) why LER decreases with inflation.

## Question No. 2:

Consider the collective risk model $S = \sum_{i=1}^{N} X_i$, in which the primary and secondary distributions are given, respectively, by

| $n$ | 0 | 1 | 2 |
|---|---|---|---|
| $\Pr(N=n)$ | 0.5 | 0.3 | 0.2 |

and

| $x$ | 1 | 2 |
|---|---|---|
| $\Pr(X=x)$ | 0.6 | 0.4 |

(a) Determine $\text{VaR}_\alpha(S) := \inf\{s \in \mathbb{R} : \Pr(S \leq s) \geq \alpha\}$ for all $\alpha \in (0,1)$.

(b) Calculate $\text{CVaR}_{0.80}(S) := \dfrac{1}{1-0.80} \displaystyle\int_{0.80}^{1} \text{VaR}_\alpha(S)\, d\alpha.$

(c) Now assume that the insurer wants to lower the $\text{CVaR}_{0.80}$ of the aggregate claims to 2.5, and to achieve that goal, it introduces a policy limit $u$ per claim. Denote the aggregate claims after the coverage modification by $\tilde{S}$. Note that

$$\tilde{S} = \sum_{i=1}^{N} (X_i \wedge u).$$

Calculate the desired $u$ such that $\text{CVaR}_{0.80}(\tilde{S}) = 2.5$.

## Question No. 3:

Consider a full insurance policy (i.e., it has a zero deductible and no policy limit) and assume its aggregate claim $S$ for the current year is given by the following collective risk model:

$$S = \sum_{i=1}^{N} X_i,$$

for which $N$ follows a Poisson distribution with mean 2, and $X_i \stackrel{d}{=} X$ follows an exponential distribution with mean 50.

(a) Suppose the premium $P$ of such a policy is determined by the expected-value premium principle with 20% loading, i.e.,

$$P = 1.2 \times \mathbb{E}[S].$$

Calculate the premium $P$.

(b) Apply normal approximation to calculate the probability that $S > 2\mathbb{E}[S]$.

(c) Due to inflation, the claim severity in the next year is estimated to increase by 10%, but the claim frequency is not impacted by inflation. In order to keep the premium $P$ unchanged for the next year, the insurer is planing to introduce an ordinary deductible $d$.

Calculate $d$.

## Question No. 4:

The probability mass function (pmf) of a zero-truncated Poisson random variable $X$ can be expressed as:

$$p_k = \frac{e^{-\lambda}}{1-e^{-\lambda}} \cdot \frac{\lambda^k}{k!}, \quad k = 1, 2, \ldots, \quad \lambda > 0.$$

(a) Show that the first two moments of a zero-truncated Poisson distribution are:

$$\mathbb{E}[X] = \frac{\lambda}{1-e^{-\lambda}} \qquad \text{and} \qquad \mathbb{E}[X^2] = \frac{\lambda + \lambda^2}{1-e^{-\lambda}}$$

(b) Establish the following relationship of the mean and variance as

$$\text{Var}[X] = \mathbb{E}[X](1 + \lambda - \mathbb{E}[X]),$$

and therefore, deduce that

$$\text{Var}[X] \leq \mathbb{E}[X].$$

**Question No. 5:**

(a) The following observations sampled from a Weibull distribution $\mathcal{W}(\alpha, \lambda)$ are given:

$$54,\ 70,\ 75,\ 81,\ 84,\ 88,\ 97,\ 105,\ 109,\ 114,\ 122,\ 125,\ 128,\ 139,\ 146,\ 153.$$

Apply the percentile matching method at 20th and 70th percentiles to estimate $\alpha$ and $\lambda$.

Note: The cumulative distribution function (cdf) of $\mathcal{W}(\alpha, \lambda)$ is

$$F_X(x) = 1 - \exp\left(-(x/\lambda)^\alpha\right), \quad x > 0.$$

(b) The following claim payment amounts from an insurance policy with a deductible of 100 are recorded:

$$20,\ 80,\ 100,\ 170,\ 200,\ 900,\ 2400.$$

Assume the ground-up loss $X$ is modeled by $\mathcal{P}(\alpha, 400)$, i.e., its pdf is given by

$$f_X(x) = \frac{\alpha\, 400^\alpha}{(x + 400)^{\alpha+1}}, \quad x > 0.$$

Compute the maximum likelihood estimate (MLE) of $\alpha$.

Hint: To illustrate, for a ground-up loss of 150, the insurer pays 50 to settle this claim.

## APPENDIX

A random variable $X$ is said to have Pareto distribution with parameters $\alpha > 0$ and $\theta > 0$, sometimes denoted by $\mathcal{P}(\alpha, \theta)$, if its cdf is expressed as

$$F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}.$$

A discrete random variable $N$ is said to belong to the $(a, b, 0)$ class of distributions if it satisfies the relation

$$\Pr(N = k) = p_k = \left(a + \frac{b}{k}\right) \cdot p_{k-1}, \quad \text{for } k = 1, 2, \ldots,$$

for some constants $a$ and $b$. Alternatively, this relation can be expressed as a linear function as

$$k \cdot \frac{p_k}{p_{k-1}} = b + ak, \quad \text{for } k = 1, 2, \ldots.$$

The initial value $p_0$ is determined so that $\sum_{k=0}^{\infty} p_k = 1$.

The Poisson distribution belongs to the $(a, b, 0)$ class of distributions with $a = 0$ and $b = \lambda$. Its mean and variance are equal:

$$\mathbb{E}[X] = \mathrm{Var}[X] = \lambda.$$